

HARMONIC ANALYSIS AND TIDAL PREDICTION

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Introduction

Changes in sea level occur over a wide temporal as well as spatial spectrum. The fastest changes in sea level are due to the minute capillary waves. The slowest changes in sea level include eustatic changes caused by the movement of tectonic plates, and, the slow rise in the global sea level due to the Greenhouse Effect. Between these two extremes we have wind waves, swell, infra-gravity waves, seiches, storm surges, tsunamis, tides, steric changes etc.

Tides are of two types, viz. astronomical and meteorological. Meteorological tides are caused by meteorological phenomena. Meteorological tides are approximately periodic in nature. Let us look at some examples of meteorological tides. The land breeze and sea breeze observed in many places is responsible for the periodic (period ≈ 1 day) fall and rise in sea level respectively. The periodic (period $\approx 1/2$ day), though small oscillation in atmospheric pressure observed in the tropics causes a periodic oscillation in the sea level. Also the melting of snows in great mountain ranges as well as the annual variability in rainfall and evaporation produce approximately periodic changes of level in estuaries of rivers.

Astronomical tides are caused by the forces of attraction exerted by the moon and the sun upon the earth and are explained on the basis of Newton's law of Gravitation. The periodic changes in the attractive forces arising from astronomical motions like the rotation of the earth about its axis, the revolution of the moon around the sun, the revolution of the earth around the sun and the precession of lunar nodes, are responsible for the different periodicities in the observed tide. In brief, astronomical tides are caused directly by astronomical phenomena. We shall restrict our attention to astronomical tides.

Newton's Law of Gravitation

Newton's law states that the force of attraction A between two bodies of mass m and M separated by distance d is given by

$$A = G m M/d^2$$

where $G (= 6.668 \times 10^{-8} \text{ cm}^3/\text{g s}^2)$ is the universal constant of gravitation.

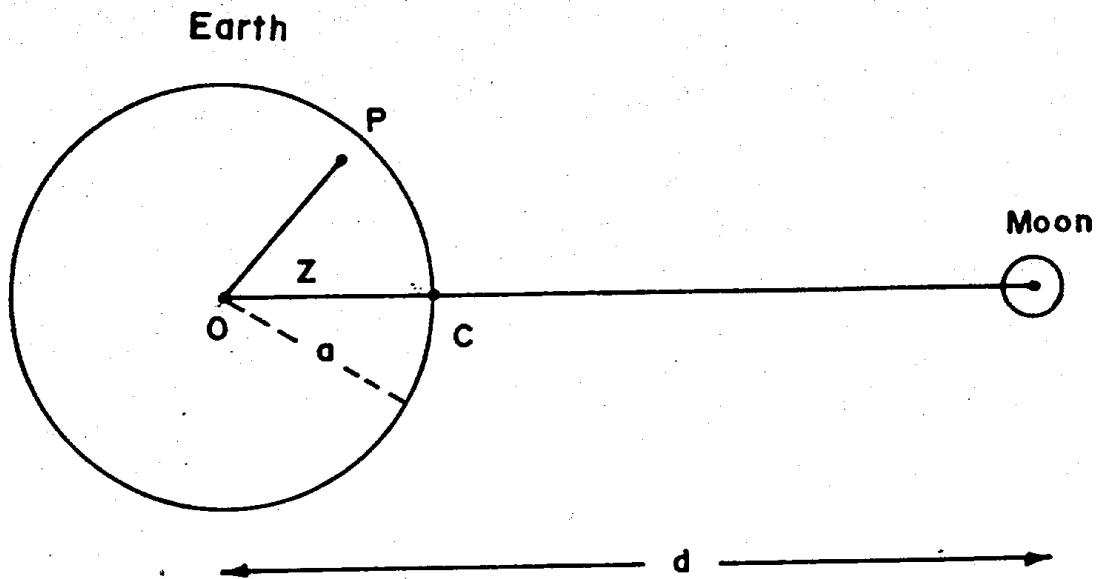


Fig. 1 Diagram for deriving the tide generating force

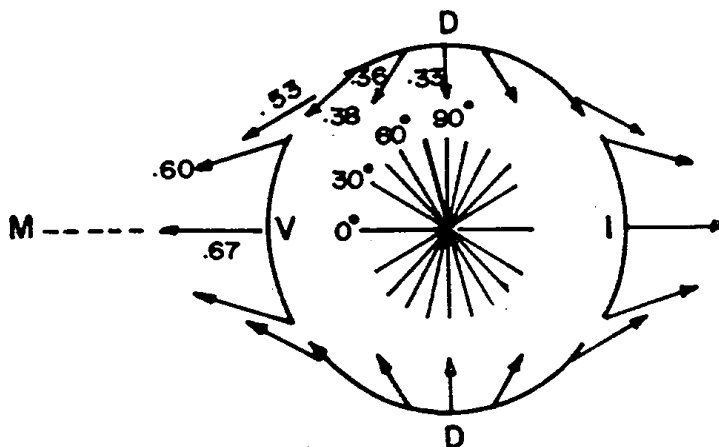


Fig. 2 Tide generating force at the equatorial cross section of the earth

Tide Generating Forces

In Fig.1, O is the centre of the earth; d is the distance from the centre of the earth to the centre of the tide generating body, the moon (say); a is the mean radius of the earth; C is the point on earth nearest to the moon; and M is the mass of the moon.

Then by Newton's Law of Gravitation the force of attraction to the moon exerted on a unit mass at C and O are $\frac{GM}{(d-a)^2}$ and $\frac{GM}{d^2}$ respectively.

The tide generating force (TGF) at any point on the surface of the earth is the difference between the attraction at that point and at the centre of the earth. Thus the TGF at C acting towards the moon is given by

$$\begin{aligned}
 F &= \frac{GM}{(d-a)^2} - \frac{GM}{d^2} \\
 &= \frac{GM}{d^2} \left[\frac{1}{\left(1 - \frac{a}{d}\right)^2} - 1 \right] \quad \dots (1)
 \end{aligned}$$

From Table 1 it may be noted that the value of a/d ($=0.01659$ for the moon) is small then as shown in Appendix A, the term in square brackets in Eq (1) may by use of Maclaurins series expansion be approximated by

$$2(a/d) + 3(a/d)^2$$

So that eq (1) reduces to

$$F = \frac{2GMa}{d^3} + \frac{3GMa^2}{d^4} \quad \dots (2)$$

If we substitute G by g, where g is the acceleration due to gravity at the earth's surface using the relation

$$g = GE/a^2$$

where E is the mass of the earth, Eq (2) may be rewritten as

$$F/g = 2(M/E)(a/d)^3 + 3(M/E)(a/d)^4 \quad \dots (3)$$

Using numerical values given in Table 1, we find from Eq (3) that the TGF at C, the point on earth nearest to the moon is composed of two parts, the part involving d^3 being the much larger part. We may then conclude that the TGF at C is directly proportional to the mass of the tide generating body and inversely proportional to the cube of its distance from the earth.

Schureman (1958) derives the following general formulae for the vertical (i.e. normal to earth's surface) and horizontal (i.e. tangential to earth surface) components of the TGF at any point P on the surface of the earth:

$$F_{v3}/g = 3 (M/E) (a/d)^3 (\text{Cos}^2 Z - 1/3) \quad \dots (4)$$

$$F_{v4}/g = 3/2 (M/E) (a/d)^4 (5 \text{Cos}^3 Z - 3 \text{Cos} Z) \quad \dots (5)$$

$$F_{a3}/g = 3/2 (M/E) (a/d)^3 \text{Sin } 2Z \quad \dots (6)$$

$$F_{a4}/g = 3/2 (M/E) (a/d)^4 \text{Sin } Z (5 \text{Cos}^2 Z - 1) \quad \dots (7)$$

where the subscripts v and a indicate the vertical and horizontal components respectively; the subscripts 3 and 4 separately indicate the parts of the TGF involving the cube and the fourth power of the parallax (a/d); and Z is the angle POC with P being any point on the surface of the earth. It may be noted that Eq (3) is a particular case of Eq. (4) and (5) that results when we put Z = 0. It may also be noted that the horizontal components given by Eq (6) and (7) both reduce to zero for Z=0.

Eq. (4) - (7) give the configuration of the TGF over the whole earth. Fig. 2 shows the TGF at the equatorial cross section of the earth. From Eq. (4) and (6) which give the order of magnitude of the vertical and horizontal components respectively of the TGF due to any tide generating body, it may be concluded that

(a) Save for the moon and the sun, the TGF of other planets and stars is negligible either because of their smaller mass or their larger distance from the earth.

(b) the TGF of the sun is about 0.5 times that of the moon, the moon thus being the more important tide generating body.

(c) the tide generating forces are small, they are less than one millionth of the force of (the earth's) gravity.

Since the vertical component of the TGF acts against the force of gravity which is a million times larger, the vertical component of the TGF plays no role in raising the tides. Although as small as the vertical component of the TGF, it is the horizontal component of the TGF that is responsible for the tides as it does not come in conflict with the force of gravity.

Equilibrium Theory of Tides

The equilibrium theory of tides assumes that the entire surface of the earth is covered by a thin layer of water and that the water instantaneously responds to the TGF.

If the water were in equilibrium solely under the action of gravity, the water surface would take the shape of a spheroid.

Under the action of gravity and the TGF of the moon (or sun) the water surface assumes the shape of a prolate spheroid with its longest axis in the direction of the moon.

Schureman (1958) derives the following formulae for the equilibrium surface:

$$h_3/a = 3/2 (M/E) (a/d)^3 (\text{Cos}^3 Z - 1/3) \quad \dots (8)$$

$$h_4/a = 1/2 (M/E) (a/d)^4 (5\text{Cos}^3 Z - 3\text{Cos} Z) \quad \dots (9)$$

where h_3 and h_4 indicate parts of the equilibrium height corresponding to the cube and the fourth power of the parallar and other variables are the same as in Eq. (4) - (7). The equilibrium heights are departures from the mean spheroidal shape.

From Eq. (8) and (9), it may be computed that, when the moon is at its mean distance from the earth, the maximum rise in level is about 36 cm and maximum fall is about 18 cm, giving a tidal range of about 54 cm. In case of the sun the tidal range is about 25 cm.

It may be noted the Eq. (8) and (9) which give the equilibrium height are identical with Eq. (4) and (5) which give the vertical component of the TGF, save for a constant factor.

If we expand Eq. (8) and (9) harmonically with separate formulae for the moon and the sun, we shall perceive the different periodicities (i.e. the constituents) in the tide with their associated equilibrium amplitudes and phases - the equilibrium amplitudes being a measure of the importance of the particular constituent. Important constituents have been given names. We have long period constituents e.g. Mf; diurnal constituents e.g. K_1 , and O_1 ; and semidiurnal constituents e.g. M_2 and S_2 .

It has been found that the observed periodicities in the tide are identical with those obtained in the harmonic development of the equilibrium tide, however the amplitudes and phases do not match.

Harmonic Analysis and Prediction of Tides

The height of the tide at any time may be predicted harmonically from known amplitudes and phases of individual constituents by the following formula (see Dronkers, 1964):

$$h(t) = H_0 + \sum_{n=1}^{NC} f_n H_n \text{Cos}(W_n t + [E_n + u_n] - g_n) \quad \dots (10)$$

in which

h = height of the tide at any time t

- t = time reckoned in hours from some initial epoch such as beginning of year of prediction
 H_0 = mean height of water level above datum used for predictions
 H_n = mean amplitude of any constituent A
 f_n = nodal correction for reducing mean amplitude H_n to year of prediction
 W_n = speed of constituent in degrees per solar hour
 $(E_n + u_n)$ = value of equilibrium argument of constituent A at Greenwich at the origin of time, i.e. when $t = 0$
 g_n = Greenwich phase of constituent A
 NC = number of constituents required for adequate description of the tide

Constituent speed W_n is synonymous with periodicity, periodicity in hours = $360/W_n$. The harmonic constants H_0 , H_n and g_n are generally available from publications of the International Hydrographic Bureau, Monaco. E_n , u_n and f_n depend solely on the origin of time and can be easily determined (See Glen, 1977 or Doodson and Warburg, 1941). Thus the only variable on the right hand side of Eq (10) is t , varying which we can determine the tidal curve. Thus the forward problem : given the harmonic constants H_n and g_n , to determine the tide - is easily solved.

We shall now discuss the inverse problem : given NH hourly observations to determine the tidal constants H_n and g_n . Three methods are available for the determination of harmonic constants. The classical approaches of Doodson (1927) uses a system multipliers for determining the constants, and was appropriate in the early days when computations had necessarily to be made by hand. With the advent of computers, and their ability for inverting large matrices, the least squares method (LSM) has become the standard method for determining tidal constants, though the response method of Munk and Cartwright (1966) is increasingly being used. We shall now briefly review the LSM.

By means of the addition formula for the cosine function, Eq (10) can be written as

$$h(t) = H_0 + \sum_{n=1}^{NC} A_n \cos W_n t + \sum_{n=1}^{NC} B_n \sin W_n t \quad \dots (11)$$

where,

$$A_n = f_n H_n \cos(E_n + u_n - g_n)$$

$$B_n = -f_n H_n \sin (E_n + u_n - g_n)$$

then if A_n and B_n are known, the tidal constants are given by

$$H_n = \frac{\sqrt{A_n^2 + B_n^2}}{f_n} \quad \dots (12)$$

$$g_n = E_n + u_n - \arctan (-B_n/A_n) \quad \dots (13)$$

If we set $t = 0$ to coincide with some time origin, say, 0000 hrs on the central day at which hourly observations are available, then Eq (11) leads to NH linear equations in which the 2NC unknown are A_n and B_n . The number of hourly observations NH and the number of constituents NC are suitably chosen such that NH is much larger than 2NC, i.e. the ratio 2NC/NH is small, so that the system of equations generated by Eq (11) is grossly overdetermined.

Let h_1, h_2, \dots, h_{NH} and $\hat{h}_1, \hat{h}_2, \dots, \hat{h}_{NH}$ respectively represent, the original series of tidal observations, and the series predicted by Eq (11). Values of (A_n, B_n) , for $n = 1, 2, \dots, NC$ are then determined by solving the normal equations obtained by

$$\text{minimizing the value of } \sum_{i=1}^{NH} (\hat{h}_i - h_i)^2$$

Once A_n and B_n are determined, the tidal constants H_n and g_n are obtained from Eqs (12) and (13). The hindcasting skill (HCS) of (A_n, B_n) is defined as follows :

$$HCS = \frac{\sum_{i=1}^{NH} (\hat{h}_i - H_0)^2}{\sum_{i=1}^{NH} (\hat{h}_i - h_i)^2} \quad \dots (14)$$

where H_0 is the mean height. HCS which takes values between 0 and 1, determines the reliability of the least squares solution (A_n, B_n) . If HCS is very close to 1, then the values of A_n, B_n are very reliable, provided that the ratio 2NC/NH is small. On the contrary if HCS is close to zero, then A_n, B_n and consequently H_n, g_n are not at all reliable.

In literature, often, instead of the Greenwich phase g_n , the local phase K_n is given. The phases g_n and K_n are related as follows :

In literature, often, instead of the Greenwich phase g_n , the local phase K_n is given. The phases g_n and K_n are related as follows :

$$g_n = K_n + pL - W_n S/15$$

where, L is the longitude of the place; S is the time meridian of the place, both reckoned as degrees west of Greenwich; $p=1, 2$ and 4 respectively for diurnal, semi-diurnal and ter-diurnal constituents; and W_n is the speed of the constituent in degrees per solar hour.

Suggested Readings

1. Doodson, A.T., 1927. *The analysis of tidal observations*. Phil. Trans. Roy. Soc., A227: 223-279.
2. Dronkers, J.J., 1964. *Tidal computations in rivers and coastal waters*. North Holland Pub. Co., Amsterdam, 518
3. Glen, N.C., 1977. *The admiralty method of tidal prediction, N.P. 159*. Int. Hydrogr. Rev., LIV(1): 73-85.
4. Munk, W.H. and Cartwright, D.E., 1966. *Tidal spectroscopy and prediction*. Phil. Trans. Roy. Soc. London, A259: 533-581.
5. Schureman, P., 1958. *Manual of harmonic analysis and prediction of tides*. U.S. Dept. of Commerce, Special Publication No. 98.

Table 1 - Astronomical data

Diameter of earth	= 12,753 km
Diameter of moon	= 3,479 km
Mass of earth	= 5.98×10^{24} kg
Mass of moon	= 7.34×10^{22} kg
Mass of sun	= 1.96×10^{30} kg
Average distance between centres of earth and moon	= 384,329 km
Average distance between centres of earth and sun	= 149,360,000 km
Universal constant of gravitation	= 6.668×10^{-8} cm ³ /gs ²

Appendix A

We are required to simplify the expression

$$\frac{1}{\left(1 - \frac{a}{d}\right)^2} - 1 \quad \dots (x)$$

where $\frac{a}{d}$ is known to be small. Let $\frac{a}{d} = x$. Then the above expression equals

$$\frac{1}{(1-x)^2} - 1 = f(x), \text{ say}$$

Since x is small, $f(x)$ may be approximated by the first few terms of its Maclaurin's series expansion.

Maclaurin's theorem states that

$$f(x) = f(0) + f^{(n)}(0) x^n \quad \dots (1)$$

where $f^{(n)}(0)$ is the value of the n th derivative of $f(x)$ at $x = 0$. Thus for

$$f(x) = (1-x)^{-2} - 1$$

we may easily see that

$$f(0) = 0$$

$$f^{(1)}(x) = 2(1-x)^{-3} \text{ so that } f^{(1)}(0) = 2$$

$$f^{(2)}(x) = 6(1-x)^{-4} \text{ so that } f^{(2)}(0) = 6$$

And substituting the above values of $f(0)$, $f^{(1)}(0)$ and $f^{(2)}(0)$ in Eq. (1), we have

$$f(x) = 2x + 3x^2 + \dots$$

where higher powers of x may be ignored as x is small. On resubstitution for x , the value of the expression (X) is therefore

$$2(a/d) + 3(a/d)^2.$$