

## Complementary Terrain/Single Beacon-Based AUV Navigation <sup>\*</sup>

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### Abstract:

This paper describes work done towards the development of advanced geophysical-based navigation systems for autonomous underwater vehicles (AUVs). The specific problem that we tackle is that of combining terrain-aided navigation (TAN) with single-beacon navigation (SBN) techniques. The resulting complementary TAN/SBN system has the potential to overcome some of the problems that arise with TAN navigation only, when an AUV undergoes motions that lead it temporarily across areas where the terrain below is not sufficiently “rich” in terms of topographic features. The key contribution of this paper is a formal analysis of the benefits of using complementary filtering, in opposition to TAN navigation only. To this effect, we exploit key tools of estimation theory, and in particular the Cramér-Rao lower bound inequality to obtain a lower bound on the minimum covariance of the estimation error that can be obtained with any unbiased estimator. For a real terrain profile we compute and compare the Cramér-Rao lower bounds for TAN only and TA/SB-based navigation. The increase in the expected performance that can be achieved with the second solution is clearly visible. The efficacy of the new solution proposed is illustrated with the help of computer simulations.

*Keywords:* Terrain-aided navigation; Single-beacon navigation; Cramér-Rao bounds; Monte Carlo methods; Particle filters.

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### 1. INTRODUCTION

The advent of autonomous underwater vehicles (AUVs) has dramatically improved the quality of the tools available for ocean data acquisition. Equipped with advanced sensor suites, these vehicles have the capability to roam the oceans freely and acquire ocean data at unprecedented scales. Central to the operation of the vehicles is the availability of good navigation and positioning systems. While at the surface, autonomous marine vehicles may resort to the Global Positioning Systems (GPS). However, this is not an option for AUVs operating submerged.

Underwater navigation has a long and rich history and has proven to be a key ingredient in the operation of submerged vehicles for scientific, commercial, and military operations. In the case of AUVs, the solutions adopted include conventional navigation using dead-reckoning methods based on inertial navigation systems (INS) complemented with discrete-time position fixes provided by acoustic baseline systems, that permit the correction of the INS inherent drifts. The solutions come with different accuracy, cost, and complexity; see e.g. Kinsey et al. (2006). Recently, with a view to cost reduction, a trend was

started on the development of single beacon navigation (SBN) systems to work in isolation or to complement the aforementioned navigation systems; Larsen (2000), Lapointe (2006), and Crasta et al. (2011). There is also a surge of interest in the development of reliable navigation systems for autonomous underwater vehicles that rely on information extracted from the surrounding environment (e.g. terrain information) without incurring in the cost of deploying artificial beacons. Map based navigation, also called Terrain-Aided Navigation (TAN), holds good potential to be used for long range missions without incurring in a cost escalate. However, to the best of our knowledge TAN has not yet realized its full potential as an integral part of underwater navigation systems.

As a contribution to the development of advanced, cost effective terrain based navigation systems, we propose a new architecture that brings together TAN and SBN. In this paper we consider a simple set-up that consists of a single echo-sounder looking downwards to measure the altitude of the AUV above the terrain, aided with measurements of the ranges between the vehicle and a single transponder (often referred to as beacon) at a known position. Our aim is to show analytically, via the computation of appropriate Cramér-Rao lower bounds (CRLB), that there is a clear advantage in complementing terrain data with range data. This is especially noteworthy when the vehicle goes temporarily over parts of the terrain that are not “sufficiently rich” in terms of topography.

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<sup>\*</sup> This research was supported in part by project ATLAS/FCT-PT (PTDC/EEA-ELC/111095/2009) that includes FEDER funds and project CO3AUVs of the EU, Grant agreement No: IST-231378. The work of the first and second authors benefited from a doctoral and post-doc grant, respectively from the Portuguese Foundation for Science and Technology (FCT).

Simulations obtained with a realistic terrain map illustrate the potential of the new solution proposed.

The paper is organized as follows. Section 2 affords the reader a brief survey of previous work on terrain-aided navigation, states the problem at hand, and describes the model used for TAN filter design. Section 3 focuses on the problem of complementary terrain/single beacon-based AUV navigation and derives the corresponding Cramér-Rao lower bounds. Section 4 describes the results of simulations aimed at showing the efficacy of the TAN/SBN filter structure. Finally, Section 5 summarizes the results and describes problems that warrant further research effort.

## 2. TERRAIN AIDED NAVIGATION: PREVIOUS WORK AND FILTER DESIGN MODEL

The techniques of geophysical based navigation rely on the solution of nonlinear estimation problems that address explicitly the availability of geophysical data (e.g. terrain and geomagnetic data). In case only terrain (bathymetric) data are used, the resulting navigation systems fall in the scope of what is commonly called Terrain Aided Navigation (TAN). In this case, and in its simplest form, local bathymetry measurements are obtained using an echo sounder and a depth cell. The data obtained are then compared with an existing bathymetric map of the area, stored on-board the underwater vehicle. Filtering techniques that merge this information with that obtained with dead reckoning data are then used to estimate the vehicle's position and velocity on-line.

The literature that illustrates practical implementations of TAN for underwater vehicles is scarce, probably due to the secrecy involved in the military systems that pioneered this type of navigation. However, there are a few reports mentioning the utilization of terrain navigation by military submarines (see e.g. Rice et al. (2000, 2004)) and some interesting publications on the development of TAN systems for autonomous underwater vehicles. An example of the latter is reported in Nygren and Jansson (2004), which describes a correlation method to match the measured bathymetry obtained with a multibeam sonar with an existing map to determine the position of an AUV. More recently, a TAN filter that uses multibeam sonar data to correct the combined INS-DVL dead-reckoning navigation drifts has been developed for the Autosub 6000 AUV Morice et al. (2009). Due to the high cost of high-accuracy INS and high-resolution sonars like multibeam echosounders, other authors explored the performance trade-offs of TAN resulting from using low-cost sensors (e.g. sonar altimeters) and lower accuracy inertial systems; Di Stefano and Mattoccia (2003).

The application of Bayesian estimation methods based on point-mass approximations of the probability distributions of interest is extensively documented in the literature on TAN. The adequacy of these methods and their superior performance relatively to parametric estimation approaches (e.g. Kalman-based) and batch-oriented matching processes like the Terrain Contour Matching (TERCOM) Ektekin (2007) is shown among others in Bergman (1999).

The determination of lower bounds for the estimation errors in TAN applications is addressed in Karlsson et al. (2003) as well as in some of the above cited publications. Interestingly enough, in Bergman (1999) and Teixeira (2007) the authors compute the posterior CRLB for the estimation problem at hand using a recursive method for a one-step predictor with explicit applications to TAN.

These two references motivated our work on the computation of the CRLB for estimation problems of TAN augmented with single beacon ranging information. An in-depth study of the subject of determination of the Cramér-Rao lower bound and the problem of super-efficiency of TAN estimators can be found in Nygren (2005).

In Teixeira (2007) and Teixeira and Pascoal (2008) the authors study geophysical based navigation systems for autonomous underwater vehicles using both terrain and geomagnetic maps. The techniques used borrow from nonlinear particle filter theory. In the TAN method developed in Teixeira (2007) the filter design model relies on the kinematic model of the vehicle with velocity measurements relative to the water being input periodically to the system from a Doppler Velocity Logger (DVL). The observations used consist of sonar range measurements that include the altitude of the vehicle relative to the seabottom and two slant ranges that allow for the determination of a local terrain gradient. The TAN estimator is a Rao-Blackwellized implementation of the sequential importance sampling with resampling (SISR) algorithm that sequentially matches the terrain-derived observations with a pre-existent bathymetric map of terrain. The method has been extended to include measurements of other geophysical variables in order to implement a Geophysical Navigation (GN) system. An important part of work in Teixeira (2007) involved modeling the acoustic sensors and the corresponding sensor errors with the objective of developing a realistic simulator that includes synthetic as well as real bathymetric maps. In fact, to avoid unrealistic simplifications of the measurement error model an algorithm was developed that simulates sonar returns from a conventional echo-sounder, taking in account beam geometry, sea-bottom reflective properties, range, and angle of incidence; see Teixeira (2007). This model was adopted to generate the measurement data used in the simulations, after appropriate processing of the real bathymetric map in figure 2.

In the course of the present work we avail ourselves of some of the results and techniques proposed in Teixeira (2007) for TAN and further expand them to explicitly include techniques that pertain to SBN. In preparation for the core work reported in the paper, we now review briefly the TAN problem statement and the model used for filter design explained there and in Teixeira and Pascoal (2008), to which the reader is referred for details.

### 2.1 Basic notation and problem statement

The following notation will be used in the sequel:  $\{I\}$  represents an inertial coordinate frame,  $\{B\}$  denotes the body-fixed frame that moves with the vehicle,  $\mathbf{p} = [x, y, z]^T$  is the position of the origin of  $\{B\}$  measured in  $\{I\}$ ,  $\lambda = [\phi, \theta, \psi]^T$  represents roll, pitch, and yaw angles that parametrize locally the orientation of  $\{B\}$  relative to  $\{I\}$ , and  $\omega = [p, q, r]^T$  represents the angular velocity of  $\{B\}$  w.r.t.  $\{I\}$ , expressed in  $\{B\}$ . We assume there is a constant current  $\mathbf{b}$  measured in  $\{I\}$ . The velocity of the AUV with respect to the water is denoted  $\mathbf{V}$ .

With the above notation, the TAN problem can be stated as follows: *Given the kinematic model of an underwater vehicle, a bathymetric map of the area of interest, and measurements of the AUVs altitude above the seabed and velocity  $\mathbf{V}$  with respect to the water, estimate the position and the velocity of the vehicle relative to  $\{I\}$  in the presence of a constant ocean current  $\mathbf{b}$ .*

### System models

For simplicity of analysis we assume that the vehicle is leveled horizontally and stabilized in roll and pitch, i.e.  $\phi = \theta = 0$  so that the angular velocity yaw rate is given by  $r = \dot{\psi}$ . The velocity vector of the current is considered constant or varying very slowly in time. Without loss of generality, we assume that the AUV maintains constant depth ( $z$ ) which can be measured with high accuracy by using a pressure cell.

From now on we treat  $z$  as an input, not as a state variable, thus reducing the dimension of the state vector; this is crucial to the performance of the Particle Filter estimator used in Teixeira (2007). Thus, the position and velocity estimation is reduced to a 2-D estimation problem. The body  $\{B\}$  and inertial reference frame  $\{I\}$  are also considered in 2-D. See Appendix A of Teixeira (2007) for a full derivation of the 3-D continuous-time model and its discrete-time counterpart. For TAN, the continuous-time design model in 2D is given by

$$\dot{\mathbf{p}} = \mathfrak{R}(\psi)\mathbf{V} + \mathbf{b} + \xi_v \quad (1)$$

$$\dot{\mathbf{b}} = \xi_b \quad (2)$$

where  $\mathbf{p} = [x, y]^T$  is the position of the vehicle,  $\mathbf{V} = [v_u, v_v]^T$  is the linear velocity relative to the water expressed in  $\{B\}$ , and  $\mathbf{b} = [b_x, b_y]^T$  is the velocity bias induced by ocean currents expressed in  $\{I\}$ . The rotation matrix  $\mathfrak{R}$  is parametrized by  $\psi$ . In the above,  $\xi_v$  and  $\xi_b$  are independent white noise processes. The heading angle  $\psi$ , the velocity vector  $V$ , and yaw rate  $r$  are used as inputs to the model. In the 2-D formulation that we will use henceforth the state vector is  $\mathbf{x} = [x, y, b_x, b_y]^T$  and the input vector is  $\mathbf{u} = [v_u, v_v, \psi, r, z]^T$ , where  $\psi$  and  $r$  are estimated using a simple complementary filter (Teixeira (2007)). The discrete-time process and measurement models for the 2D TAN problem can now be written as

$$\mathbf{x}_{k+1} = F\mathbf{x}_k + G_u\mathbf{u}_k + \zeta_k \quad (3)$$

$$\mathbf{y}_k = h(\mathbf{x}_k) + \eta_k(\mathbf{p}_k), \quad (4)$$

where  $h(\mathbf{x}_k)$  represents the altitude of the vehicle above the terrain,  $\zeta_k$  is process noise and  $\eta_k$  is measurement noise; the latter is a function of position  $\mathbf{p}_k$ . In the above,  $F$  and  $G_u$  are matrices of appropriate sizes that depend on the sampling period  $T$  and  $G_u$  is a function of the input  $u$ . Notice that

$$h(\mathbf{x}_k) = m(x, y) - z \quad (5)$$

where  $m(x, y)$  is the terrain elevation (bathymetric) map measured from the surface and  $z$  is the depth of operation of the AUV. Notice also that  $\eta_k(\mathbf{p})$  captures both measurement noise and map errors, including those that arise in the process of smoothing that same map. For our purposes,  $\eta_k(\mathbf{p})$  will be modeled as Gaussian white noise. We note, however, that the parameters of the gaussian distribution that models the sonar range error may change, depending on the acoustic properties of the sea-bottom material. A more realistic model is proposed in Gustafsson et al. (2001) by using a sum of gaussians. See also the uncertainty model derived in Teixeira (2007) that allows the parametrization of sea-bottom properties, including bottom surface roughness and reflectance parameters.

### 3. COMPLEMENTARY TERRAIN/SINGLE BEACON-BASED AUV NAVIGATION

In spite of its potential to yield a reliable method of navigation for long range operations, TAN suffers from a

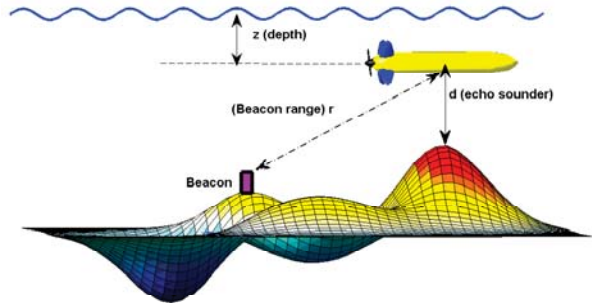


Fig. 1. Complementary Terrain and Single Beacon-Based Navigation

major drawback: in the presence of a flat seabed, it may fail to generate good navigation data due to the lack of sufficiently rich terrain information. This problem is especially acute in the case of small and low cost vehicles, which cannot carry expensive Doppler and Heading/Attitude Reference units and are therefore incapable of doing precise dead reckoning while navigating over a featureless terrain. In such cases, navigation of the vehicle may suffer from large position errors, and these errors may grow unbounded. It is for this reason that we propose a new architecture that brings together TAN and SBN whereby a TAN algorithm is also fed with the ranges between the marine robot and an underwater transponder installed at a known location. Our aim is to show analytically, via the computation of appropriate Cramér-Rao lower bounds, that there is a clear advantage in complementing terrain data with range data.

#### 3.1 Single Beacon Navigation

Single beacon navigation has been the subject of increasing attention in the past few years. From a theoretical viewpoint, the topic raises interesting and challenging theoretical issues. From a practical standpoint, it opens not yet fully exploited possibilities for the development of affordable positioning systems. The reader is referred to Crasta et al. (2011) and Gadre and Stilwell (2004) for a brief introduction to the problem at hand. It is by now well known that single beacon navigation techniques may yield observability of the the states of the kinematic model of autonomous marine vehicles for certain classes of inputs. In Crasta et al. (2011), for example, it is shown that the model state is observable even in the presence of unknown currents if the vehicle undergoes motions at a constant turning rate. In the literature, most of the observability analysis is done in a deterministic setting and aims to establish whether the model is observable or not. Not much work has been reported on techniques to assess the level of observability. In this respect, it seems worthwhile to exploit the possible connections between the concept of observability in a non-linear setting and tools available from classical estimation theory, namely the use of the Fisher information matrix (FIM) and the corresponding Cramér-Rao lower bound (CRLB) Inequality. This is because the latter sets a bound on the best performance attainable with any unbiased vehicle position estimator. The relationship between FIM and observability has not been addressed clearly and needs a rigorous study, see Jauffret (2007). One of the main limitations from which SBN suffers is the fact that the measurement noise increases with distance, and the information depends on the location of the beacon. Nevertheless, Single Beacon Navigation has very good potential to be used in the field and to lead to simple, cost-effective navigation solutions.

### 3.2 TAN/SBN

Intuitively, it is reasonable to expect that complementary Terrain/Single Beacon-Based navigation may alleviate the problem that arises when a vehicle goes temporarily over a flat terrain. In fact, by incorporating into the navigation system data on the distance between the vehicle and a fixed transponder, it may be possible to bound the position estimation errors until the vehicle enters another area where the terrain is again sufficiently exciting. This statement, however, needs formal justification. To this effect, in the work reported here we addressed the problem of quantifying the best possible accuracy in the position estimates that can be obtained with any unbiased estimator in the cases of: i) TAN, ii) SBN, and iii) TAN/SBN. The tools that we use borrow from estimation theory and lead to the computation of so-called Cramér-Rao lower bounds for the three types of navigation systems considered. For background material on the computation of the posterior CRLB for general nonlinear filtering problems the reader is referred to Tichavsky et al. (1998), Taylor (1979), Kerr (1989) and the references therein. Related work with explicit applications to TAN can be found in Teixeira (2007) and Bergman (1999). These two references motivated much of the work that we report here, including techniques to explicitly compute the CRLB for estimation problems involving dynamic systems. Stated in very simple terms, and trading off rigor for the need to intuitively state what the key limitations of TAN are, the references show clearly how the CRLB decreases when the terrain is sufficiently exciting, where the terrain excitation is measured locally in terms of the absolute value of its gradient.

### 3.3 The Cramér-Rao lower bound (CRLB)

For a navigation system based on TAN complemented with SBN, the observation model is similar to the one in (4), with

$$\mathbf{y}_k = [h_1(\mathbf{x}_k) \quad h_2(\mathbf{x}_k)]^T + \eta_k(\mathbf{p}_k), \quad (6)$$

where  $h_1(\cdot)$  is a nonlinear function that represents the measurement model of the echo-sounder (altimeter) installed in the vehicle. This model embeds the map of the terrain that has no particular structure. The model for the range measurements relative to a single transponder is embodied in the smooth function  $h_2(\cdot)$ . The measurement error is represented by  $\eta_k(\mathbf{p}_k) = [\eta_{1k}(\mathbf{p}_k), \eta_{2k}(\mathbf{p}_k)]^T$ , where  $\eta_{1k}(\mathbf{p}_k)$  models the errors in the echo-sounder measurements while  $\eta_{2k}(\mathbf{p}_k)$  models the errors in the measurements of the ranges to the transponder; see Chapter 4 of Teixeira (2007) for more details on the sonar model. Without any loss of generality, it can be assumed that the transponder is placed at  $[0 \quad 0 \quad z]^T$ , where  $z$  is the constant depth maintained by the AUV. Thus, the second measurement equation becomes

$$h_2(\mathbf{x}_k) = \|\mathbf{p}_k\|, \quad (7)$$

The fact that the standard deviation of range measurement increases with distance from the transponder is captured by

$$\eta_{2k} = (1 + \gamma \|\mathbf{p}_k\|)\chi, \quad (8)$$

where  $\gamma$  is a constant parameter and  $\chi$  is white gaussian noise with a constant standard deviation and zero mean (see Alcocer (2009)). The noise signals in the model are assumed to be mutually independent and white, with distributions

$$\zeta_k \sim p_{\zeta_k}(\mathbf{x}_k) = \mathcal{N}(0, \mathbf{Q}_k), \text{ and} \quad (9)$$

$$\eta_k \sim p_{\eta_k}(\mathbf{x}_k) = \mathcal{N}(0, \mathbf{R}_k), \quad (10)$$

where  $\mathbf{Q}_k$  and  $\mathbf{R}_k$  are covariance matrices that may depend on the states. In the above, the noises are assumed to be zero mean only for the sake of simplicity in the computation of CRLB. In reality, these noises are not necessarily zero mean; this is true for the case of TAN measurements obtained with a conventional echo-sounder, where the mean value of the measurement error depends on the local terrain gradient (Teixeira (2007)). We now make the following assumptions: the state of the system,  $\mathbf{x}_k$  is considered to be a Markov process such that

$$p(\mathbf{x}_k) = p(\mathbf{x}_0) \prod_{i=1}^k p(\mathbf{x}_i | \mathbf{x}_{i-1}) \quad (11)$$

applies with  $p(\mathbf{x}_0)$  denoting the prior distribution for the state at  $k = 0$ ; the observations are conditionally independent given the states, i.e.,

$$p(\mathbf{Y}_k | \mathbf{X}_k) = \prod_{i=1}^k p(\mathbf{y}_i | \mathbf{x}_i) \quad (12)$$

where  $\mathbf{X}_k$  represents the sequence of states  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$  and  $\mathbf{Y}_k$  represents the sequence of observations  $\mathbf{y}_1, \dots, \mathbf{y}_k$ . The prior distribution of position is

$$\mathbf{x}_0 \sim p_0(\mathbf{x}_0) = \mathcal{N}(\bar{\mathbf{x}}_0, \mathbf{P}_0).$$

Given the assumptions on the noise, the prior distribution for  $\mathbf{X}_k$  is

$$p(\mathbf{X}_k) = p(\mathbf{x}_k | X_{k-1}) \dots p(\mathbf{x}_1 | \mathbf{x}_0) p(\mathbf{x}_0), \quad (13)$$

$$= p(\mathbf{x}_0) \prod_{i=1}^k p(\mathbf{x}_i - F\mathbf{x}_{i-1} - G u_{i-1}), \quad (14)$$

where  $u_i$  is the input vector represented in (3) that includes the linear velocity and the heading angle. In view of the main assumptions made before, it is straightforward to compute the likelihood function

$$p(\mathbf{Y}_k | \mathbf{X}_k) = \prod_{i=1}^k p(\mathbf{y}_i - h(\mathbf{x}_i)). \quad (15)$$

Recall now the definition of the Fisher information matrix (FIM)  $J(\theta)$  that arises in the estimation of a parameter  $\theta$  given related observations embodied in a stochastic process  $\zeta$  (Van Trees (2001)):

$$J(\theta) = -E \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \log p_{\zeta}(\zeta; \theta) \right) \right] \quad (16)$$

In what follows we identify  $\theta$  and  $\zeta$  with the extended state  $\mathbf{X}_k$  and measurement  $\mathbf{Y}_k$ , respectively. In order to compute sequentially the information submatrix, we decompose  $\mathbf{X}_k = [\mathbf{X}_{k-1}^T, \mathbf{x}_k^T]^T$ . Following the formulation introduced in Tichavsky et al. (1998) we have

$$J(\mathbf{X}_k) = \begin{bmatrix} A_k & B_k \\ B_k^T & C_k \end{bmatrix} = \quad (17)$$

$$\begin{bmatrix} E \left[ -\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \log p(\mathbf{X}_k, \mathbf{Y}_k) \right] & E \left[ -\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \log p(\mathbf{X}_k, \mathbf{Y}_k) \right] \\ E \left[ -\Delta_{\mathbf{x}_k}^{\mathbf{x}_{k-1}} \log p(\mathbf{X}_k, \mathbf{Y}_k) \right] & E \left[ -\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{X}_k, \mathbf{Y}_k) \right] \end{bmatrix} \quad (18)$$

where  $\Delta_x^y = \nabla_x \nabla_y^T$  and  $\nabla_x = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]^T$ . The recursive expression for  $J(\mathbf{x}_k)$  is derived in Tichavsky et al. (1998). We denote  $J(\mathbf{x}_k)$  as  $J_k$  for brevity. The expression for  $J_k$  is given by

$$J_k = D_{k-1}^{22} - D_{k-1}^{21} (J_{k-1} + D_{k-1}^{11})^{-1} D_{k-1}^{12} \quad (19)$$

where,

$$\begin{aligned}
D_k^{11} &= -E \left\{ \nabla_{\mathbf{x}_k} [\nabla_{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)]^T \right\} \\
D_k^{21} &= -E \left\{ \nabla_{\mathbf{x}_k} [\nabla_{\mathbf{x}_{k+1}} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)]^T \right\} \\
D_k^{12} &= (D_k^{21})^T \\
D_k^{22} &= -E \left\{ \nabla_{\mathbf{x}_{k+1}} [\nabla_{\mathbf{x}_{k+1}} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)]^T \right\} \\
&\quad - E \left\{ \nabla_{\mathbf{x}_{k+1}} [\nabla_{\mathbf{x}_{k+1}} \log p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})]^T \right\},
\end{aligned}$$

with  $J_0$  computed from the initial density  $p(x_0)$  as  $J_0 = E \{ [\nabla_{\mathbf{x}_0} \log p(\mathbf{x}_0)] [\nabla_{\mathbf{x}_0} \log p(\mathbf{x}_0)]^T \}$ . If the initial distribution is Gaussian:  $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{x}_0; \bar{\mathbf{x}}_0, P_0)$ , then it is easy to show that  $J_0 = P_0^{-1}$ . For a linear process model (3) and additive Gaussian noise with zero mean ((9), (10)) we have,

$$D_k^{11} = F^T Q_k^{-1} F \quad (20)$$

$$D_k^{12} = -F^T Q_k^{-1} \quad (21)$$

$$D_k^{22} = Q_k^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1} \quad (22)$$

where  $R_k$  is the measurement noise intensity matrix (see (10)) and  $H_k = [\nabla_{\mathbf{x}_k} h^T(\mathbf{x}_k)]^T$  is the Jacobian of  $h(\mathbf{x}_k)$  evaluated at the true state  $\mathbf{x}_k$ , related to the local terrain gradient represented by  $H_k^T$ . Thus,  $J_k$  can be written as

$$\begin{aligned}
J_k &= Q_{k-1}^{-1} + H_k^T R_k^{-1} H_k \\
&\quad - (F^T Q_{k-1}^{-1})^T (J_{k-1} + F^T Q_{k-1}^{-1} F)^{-1} F^T Q_{k-1}^{-1} \quad (23)
\end{aligned}$$

and  $H_k$  is given by

$$H_k = \begin{bmatrix} H_{1k} \\ H_{2k} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x}_k)}{\partial x_k} & \frac{\partial h_1(\mathbf{x}_k)}{\partial y_k} & \frac{\partial h_1(\mathbf{x}_k)}{\partial b_{x_k}} & \frac{\partial h_1(\mathbf{x}_k)}{\partial b_{y_k}} \\ \frac{\partial h_2(\mathbf{x}_k)}{\partial x_k} & \frac{\partial h_2(\mathbf{x}_k)}{\partial y_k} & \frac{\partial h_2(\mathbf{x}_k)}{\partial b_{x_k}} & \frac{\partial h_2(\mathbf{x}_k)}{\partial b_{y_k}} \end{bmatrix}.$$

In the first two columns of the above matrix,  $\frac{\partial h_1(\mathbf{x}_k)}{\partial x_k}$  and  $\frac{\partial h_1(\mathbf{x}_k)}{\partial y_k}$  are the components of the local gradient of the terrain represented in the map with respect to the 2D coordinates. It is hard to attach some physical meaning to the gradient with respect to the velocity of ocean currents. Following common practice, in our simulations we consider that the derivative represented in the last two columns of the matrix are zero. The inverse of matrix  $R_k$  is given by

$$R_k^{-1} = \begin{bmatrix} \frac{1}{\sigma_{1k}^2} & 0 \\ 0 & \frac{1}{\sigma_{2k}^2} \end{bmatrix},$$

where  $\sigma_{1k}^2$  is the intensity of the sonar measurement noise and  $\sigma_{2k}^2$  is the intensity of the error  $\eta_{2k}$  in the range measurements.

The Fisher information matrix represents the information content that can be extracted from the measurements with this particular system. In essence, this information is defined by the gradients of the measurement. Close examination of (23) also reveals that the FIM mainly depends on the term  $S_k = H_k^T R_k^{-1} H_k$ . If the gradient  $H_k^T$  undergoes discontinuities or if the local slope of the terrain is very large,  $S_k$  will change sharply, resulting in abrupt changes of  $J_k$ . This shows that the FIM is sensitive to local gradients and it is said to have the property of *locality* (see Frieden and Soffer (1995)). Notice that in case of terrain-based navigation the gradient of the measurement function corresponds to the gradient of the terrain represented in

the map. As such, there are undesirable effects in terms of filter efficiency that can arise by using a map with high spatial variability induced by noisy measurements if such noise is not properly modeled by the corresponding error model. This topic will be revisited in the next section. For TAN/SBN, the matrix  $S_k$  can be decomposed into a sum given by

$$S_k = S_{k_{TAN}} + S_{k_{SB}} \quad (24)$$

where

$$S_{k_{TAN}} = \frac{1}{\sigma_{1k}^2} [\nabla_{\mathbf{x}_k} h_1^T(\mathbf{x}_k)] [\nabla_{\mathbf{x}_k} h_1^T(\mathbf{x}_k)]^T = \frac{1}{\sigma_{1k}^2} H_{1k}^T H_{1k} \quad (25)$$

and

$$S_{k_{SB}} = \frac{1}{\sigma_{2k}^2} [\nabla_{\mathbf{x}_k} h_2^T(\mathbf{x}_k)] [\nabla_{\mathbf{x}_k} h_2^T(\mathbf{x}_k)]^T = \frac{1}{\sigma_{2k}^2} H_{2k}^T H_{2k}. \quad (26)$$

In (25) and (26),  $\text{tr}(S_{k_{TAN}}) \geq 0$  and  $\text{tr}(S_{k_{SB}}) \geq 0$ . Thus, it can be concluded that

$$\text{tr}(S_k) \geq \text{tr}(S_{k_{TAN}}) \text{ and } \text{tr}(S_k) \geq \text{tr}(S_{k_{SB}}) \quad (27)$$

The expression for  $S_k$  above is used to compute  $J_k$ , from which the CRLB matrix follows by inverting it. The minimum covariance of the estimation error for  $\hat{\mathbf{X}}_k$  that can be obtained with any unbiased estimator is embodied in  $J_k$  and therefore in the CRLB matrix  $J_k^{-1}$ . Stated in simple terms, the larger the trace of  $J_k$ , the smaller the covariance is. Equation (27) shows that the trace of  $J_k$  will always be greater when the single beacon is used to complement TAN than in the case of TAN navigation only, thus showing clearly the advantage of using TAN/SBN. Actually, it can be shown from the above that adding new independent measurements characterized by non-zero gradients will contribute to reducing the CRLB further. As such, complementing TAN with SBN provides complementary information from two independent sources and allows for a reduction of the CRLB. Using computations similar to those presented above we can show that adding another echo sounder will also increase the information for estimating the states. However, the combination of two echo sounders will still not provide a solution to the flat terrain or the symmetric map problem. In the work reported here, the expression for  $J_k$  was instrumental in assessing the efficacy of TAN, SBN, and TAN/SBN and drawing comparisons among them. However, the importance of having such an expression goes well beyond the theoretical results that clearly show the advantage of complementing TAN with SBN. In fact, it affords a system designer the information required to evaluate the navigational performance that can be achieved as a function of the trajectories planned for an AUV, the type of terrain encountered, and the position of the underwater acoustic transponder.

#### 4. SIMULATION SETUP AND RESULTS

In what follows, using computer simulations corresponding to a realistic mission scenario we compute the evolution of the CRLB associated with the problem of estimating the position of an AUV based on terrain and single transponder measurement data. Before we do this, however, we must call attention to an often neglected fact, explained as follows. In the computation of the CRLB, the map of the terrain plays a fundamental role for, if the terrain is very exciting, then the CRLB predicts that TAN has the potential to perform extremely well. Notice however that a terrain map must be obtained from data acquired at sea. For this reason, the unprocessed map will reflect the mixed contribution of both the ruggedness of the terrain (as if measured by an ideal altitude sensor) and

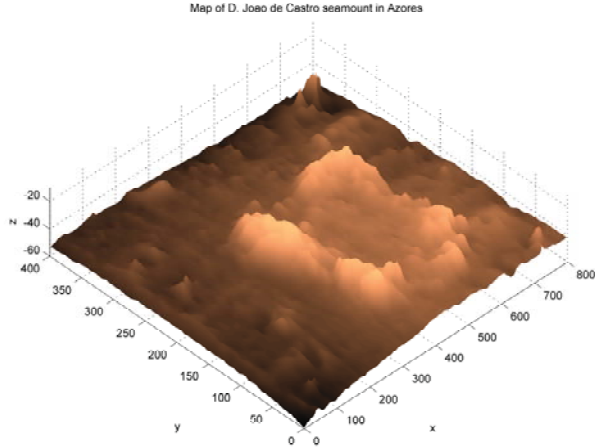


Fig. 2. Topography of the D. Joao de Castro Seamount, Azores

the deviations from reality caused by measurement noise. Clearly, measurement noise may yield a seemingly exciting terrain that does not correspond to reality and that will, in a hasty theoretical analysis, lead to CRLBs that are artificially low. Stated differently, in an unprocessed map it will be hard or virtually impossible to distinguish between actual high frequency spatial variations in the map and the spatial excitation that is artificially introduced by sensor noise. Dealing with this issue is not easy, for it requires that a good model be available for the sensors used to acquire bathymetric data. Such a model is thoroughly described in Teixeira (2007), where the author shows that the variance of the measurement noise associated with echosounder measurements depends on the distance from the sea floor and the measurement bias depends on the local terrain gradient.

The simulation setup is similar to the case of TAN only navigation described in Teixeira (2007) and Teixeira and Pascoal (2008); However, for simplicity we use only one down looking sonar (an altimeter) for TAN (see figure 1).

For the sake of brevity, we eschew the problem of deriving from actual terrain data the two items that are essential for navigation: i) a bathymetric map, and ii) a description of the noise associated with altitude measurements. Here, we simply start with an actual map obtained during a mission at sea, smooth it by removing high frequency spatial variations, and use statistics for the measurement noise that are compatible with the models exploited in Teixeira (2007). The map used for the simulation shows the topography of the D. Joao de Castro sea mount in the Azores, see figure 2. Figure 3 shows the smoothed map that was obtained from it. In the simulations, the AUV trajectory is an arc starting from a low excitation area in the map and proceeding to a higher excitation location, see figure 6. Details of the simulation parameters are shown in table 1. The posterior CRLBs, computed using the recursive method exposed in the previous section for TAN, SBN, and TAN/SBN are shown along with their RMS errors in figure 4 and 5 for the estimates of position and ocean currents, respectively. Here, it is clearly visible the improvement in performance that can be expected when TAN is aided with single-beacon measurements, especially when the terrain lacks sufficient excitation. The average over 100 MC runs of position and velocity estimates for the three configurations are shown in figure 6 and 7, respectively.

In this paper we do not address the problem of designing

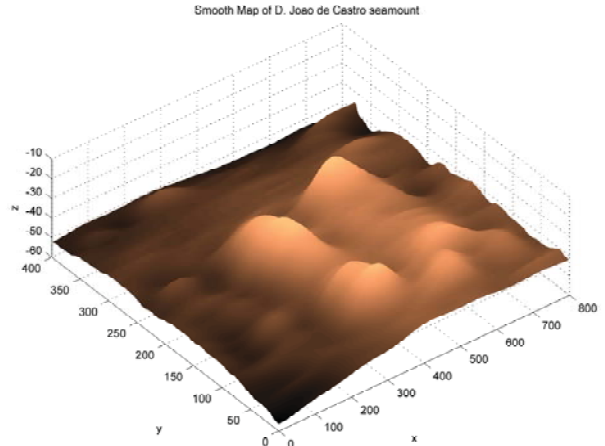


Fig. 3. Smoothed Map

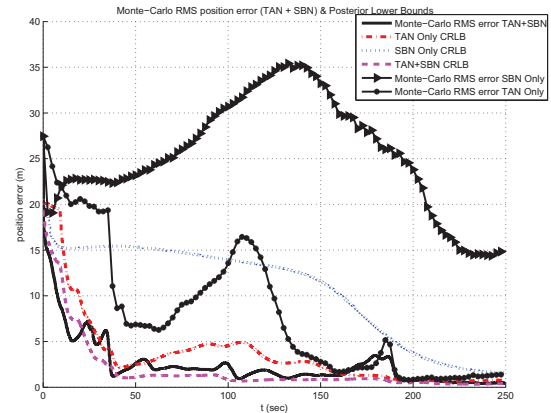


Fig. 4. Magnitude of Position Error: Posterior CRLB and Monte-Carlo RMS

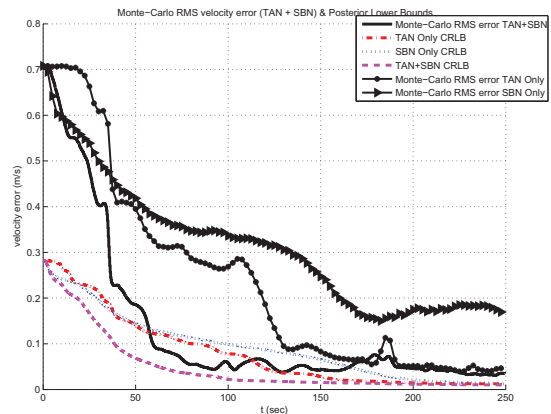


Fig. 5. Ocean Current Speed Error: Posterior CRLB and Monte-Carlo RMS

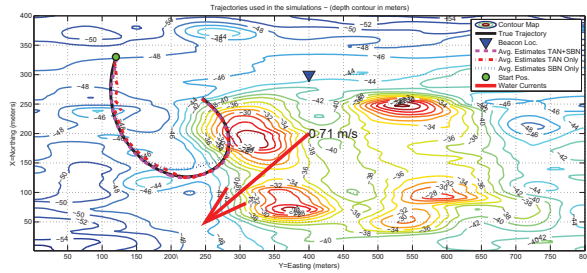


Fig. 6. Average Position Estimate

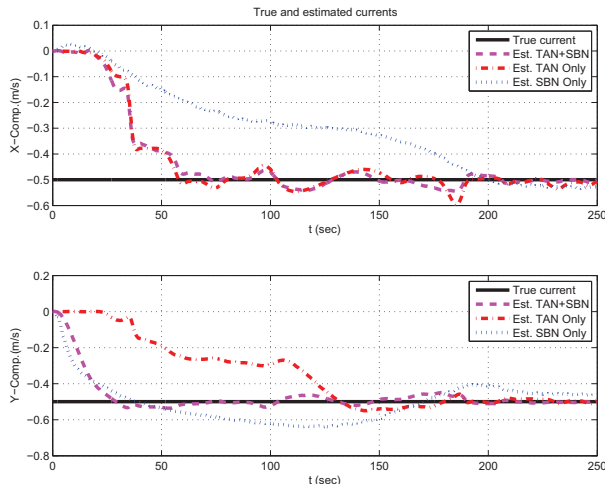


Fig. 7. Average components of the ocean current velocity vector

Parameter	Value
Filter sampling time	0.25 s
Sonar effective beam-width (downward looking)	10 deg
Vehicle speed (inertial)	2 m/s
Ocean current velocity	$[-0.5 \ -0.5]^T$ m/s
Process noise ( $Q_k$ ) covariance	$diag([0.1^2 \ 0.1^2 \ 1e-6^2 \ 1e-6^2])$
Initial Covariance	$diag([20^2 \ 20^2 \ 0.2^2 \ 0.2^2])$
Number of particles for filter	1500
Number of extra particles	500
Beacon Position	$[300 \ 400]^T$
Number of MC Runs	100

Table 1. Simulation parameters

a specific filter for TAN/SBN. However, this can be done by exploiting the circle of ideas exposed in Teixeira (2007) and Teixeira and Pascoal (2008), leading to a particle filter structure. For the sake of clarity, Figure 8 shows that the evolution of the particles for a single simulation of a TAN/SBN particle filter. Notice how at the beginning of the manoeuvre (when the available terrain information is still very low) the particles confine themselves to an arc of the circumference centred at the transponder; later in time, the cloud of particles concentrates progressively to nearly a point as soon as the richness in the terrain information increases.

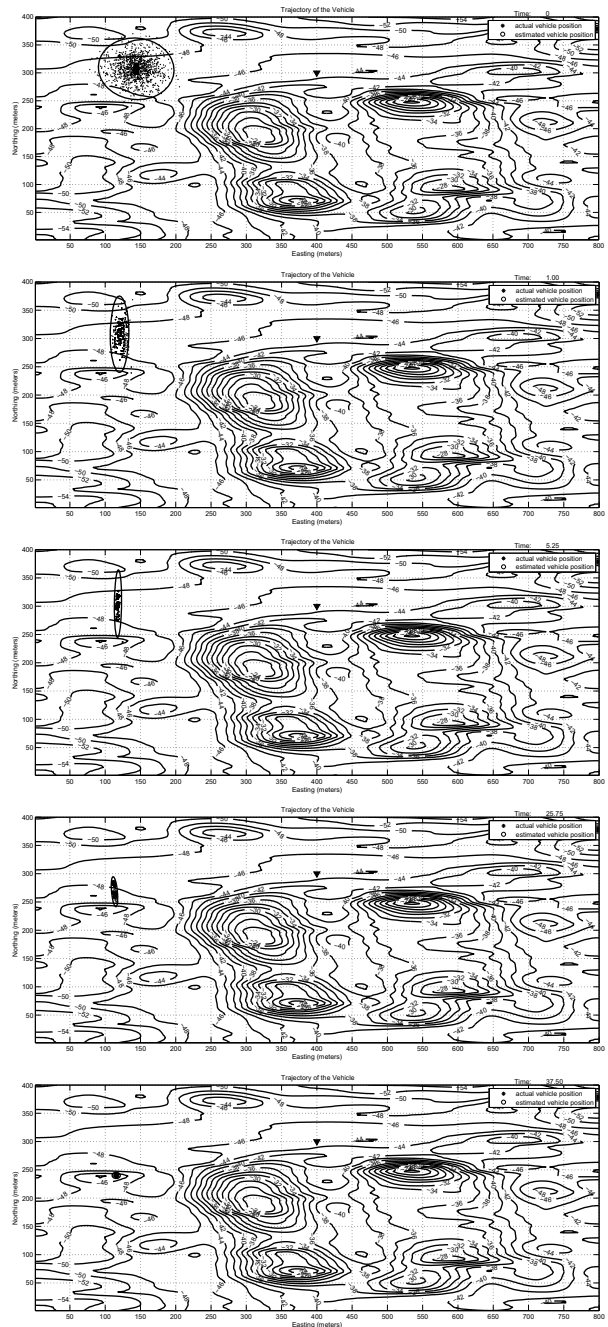


Fig. 8. TAN/SBN : particle spreading at different instants of time

## 5. CONCLUSIONS

This paper summarized work done towards the development of advanced geophysical navigation systems for autonomous marine vehicles. Namely, development and assessment of the performance that can be obtained with complementary Terrain-Aided / Single Beacon Aided Navigation. The key contribution of this paper was a formal analysis of the benefits of using complementary filtering, in opposition to TAN navigation only. To this effect, we exploited key tools of estimation theory, and in particular the Cramér-Rao lower bound inequality that yields a lower bound on the minimum covariance of the estimation

error that can be obtained with any unbiased estimator. For a real terrain profile we computed and compared the Cramér-Rao lower bounds for TAN only and TA/SB-based navigation. The increase in the expected performance that can be achieved with the second solution was clearly visible. Together with previous results obtained for pure TAN, this paper advanced a new navigation strategy that yields potential to transition from the laboratory to the real world. Implementation and field testing of the algorithms proposed for TAB/SBN have already taken place; the results will appear elsewhere. Future work will focus on cooperative TAN/SBN navigation for groups of multiple vehicles.

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