

Estimation of the Age of Exploitation at a Given Fishing Mortality

According to the concept of eumetric fishing (Beverton and Holt, 1957) there is a particular age of exploitation or selection of gear which for a fish population will maximise the yield at a given fishing mortality rate (F). Apart from reading approximately the best age of exploitation for a given F from the eumetric fishing curve or from the equilibrium yield curve, no accounts are known to me giving a more precise method of estimating this important parameter. This note therefore gives a general algebraic solution of estimating the required age of exploitation.

The equation for yield in weight (Yw) developed by Beverton and Holt for a population in a steady state is

$$Yw = FRW_{\infty} e^{-M(t_p - t_0)} \sum_{n=0}^3 \frac{\Omega_n e^{-nk(t_p - t_0)}}{z + nk} \left[1 - e^{-(Z + nk)(t_{\lambda} - t_p)} \right] \dots (1)$$

where F = instantaneous fishing mortality rate, M = instantaneous natural mortality rate, Z = instantaneous total mortality rate (i.e. F + M), R = number of recruits entering the fishing ground, W_{∞} = asymptotic weight, K = rate of deceleration in growth increments, t_0 = arbitrary age at zero length, t_p = age at which the fish enter the fishing ground, t_p' = age at which the fish is first fully exploited, and t_{λ} = maximum exploitable age. In the above equation the growth in weight of the individual fish is isometric, i.e. the value of the exponent, γ , in the equation describing the length-weight relationship is 3.

The age of exploitation maximising the yield at a given F can be calculated by differentiating the equation 1 with respect to t_p' , equating to zero, and finally solving it for t_p' . But the resulting equation can be applied only when the growth is isometric. However, when the growth exponent is not 3, yield can be calculated accurately using the method of Jones (1957) or of Paulik and Gales (1964), but these models cannot be differentiated easily with respect to t_p' . Krishman Kutty (1968) gives a modified version of the Beverton and Holt equation from which the yield can be calculated fairly accurately when growth is not isometric. This equation states

$$Yw = FRW_{\infty} e^{-M(t_p - t_0)} \sum_{n=0}^{d-1} (-1)^n \frac{\Omega_n e^{-nk(t_p - t_0)}}{Z + nk} \left[1 - e^{-(Z + nk)(t_{\lambda} - t_p)} \right] \dots (2)$$

where Ω_n takes successively the values of $1, \frac{\gamma}{1!}, \frac{\gamma(\gamma-1)}{2!},$

$$\dots \frac{\gamma(\gamma-1)(\gamma-2)\dots(\gamma-(d-2))}{(d-1)!}$$

and $d =$ number of binomial terms obtained by expanding the equation for weight at time $t,$

$$W_t = W_\infty [1 - e^{-k(t-t_0)}]^\gamma \dots (3).$$

Equation 2 is differentiable and hence by differentiating 2 instead of 1 a general solution for calculating the t_p can be obtained for all values of $\gamma.$ When γ is a positive integer the upper value $(d - 1)$ in the summation term, $\sum_{n=0}^{d-1}$ in equa-

tion 2 will be equal to the value of $\gamma.$ But when γ is not a positive integer, equation 3 will not terminate on expansion and d in equation 2 will assume a value equal to the number of binomial terms which would be sufficient for obtaining a fairly accurate estimate of yield.

Differentiating equation 2 with respect to $t_p,$

$$\begin{aligned} \frac{dYw}{dt_p} = & FRW_\infty e^{-M(t_p-t_0)} \left[\frac{1}{Z} \cdot -e^{-Z(t_\lambda-t_p)} \cdot Z - \frac{\gamma_{(1)} e^{-k(t_p-t_0)}}{Z+k} \times \right. \\ & -e^{-(Z+k)(t_\lambda-t_p)} \cdot (Z+k) + \frac{\gamma_{(1)} k e^{-k(t_p-t_0)}}{Z+k} \left\{ 1 - e^{-(Z+k)(t_\lambda-t_p)} \right\} \\ & + \frac{\gamma_{(2)} e^{-2k(t_p-t_0)}}{Z+2k} \cdot -e^{-(Z+2k)(t_\lambda-t_p)} \cdot (Z+2k) \\ & - \frac{\gamma_{(2)} 2k e^{-2k(t_p-t_0)}}{Z+2k} \left\{ 1 - e^{-(Z+2k)(t_\lambda-t_p)} \right\} \dots \left. \right] + \\ & \left[\frac{1}{Z} \left\{ 1 - e^{-Z(t_\lambda-t_p)} \right\} - \frac{\gamma_{(1)} e^{-k(t_p-t_0)}}{Z+k} \left\{ 1 - e^{-(Z+k)(t_\lambda-t_p)} \right\} \right. \\ & \left. + \frac{\gamma_{(2)} e^{-2k(t_p-t_0)}}{Z+2k} \left\{ 1 - e^{-(Z+2k)(t_\lambda-t_p)} \right\} \dots \right] \times \\ & FRW_\infty e^{-M(t_p-t_0)} \cdot -M \dots (4) \end{aligned}$$

where $\gamma_{(1)}, \gamma_{(2)},$ etc. are respectively $\frac{\gamma}{1!}, \frac{\gamma(\gamma-1)}{2!},$ etc. Putting $\frac{dYw}{dt_p} = 0,$ cancelling the common factor, $FRW_\infty e^{-M(t_p-t_0)}$ and after rearranging and simplifying, equation 4 is reduced to

$$e^{Z(t_\lambda - t_p)} \left[\frac{M}{Z} - \frac{\gamma(1)(M+k)}{Z+k} e^{-k(t_p - t_0)} + \frac{\gamma(2)(M+2k)}{Z+2k} e^{-2k(t_p - t_0)} \dots \right]$$

$$= -\frac{Z-M}{Z} + \frac{\gamma(1)(Z-M)}{Z+k} e^{-k(t_\lambda - t_0)} - \frac{\gamma(2)(Z-M)}{Z+2k} e^{-2k(t_\lambda - t_0)} + \dots (5).$$

The factor (Z-M) or F being common in all terms of RHS and the sign of all the terms alternates, equation 5 can be written as

$$e^{Z(t_\lambda - t_p)} \sum_{n=0}^{d-1} (-1)^n \frac{\gamma(n)(M+nk)}{Z+nk} e^{-nk(t_p - t_0)} = -F \sum_{n=0}^{d-1} (-1)^n \frac{\gamma(n)e^{-nk(t_\lambda - t_0)}}{Z+nk} \dots (6).$$

The exact solution of equation 6 is not easy since t_p appears in exponential terms of different powers. However, it can be calculated fairly accurately by successive approximation first by assigning two values to t_p and getting two estimates of LHS of equation 6 so that one is more and the other is less than the RHS value. Then by linear interpolation the first approximate value of t_p , say t_{p1} is calculated. An estimate of LHS is made again after inserting the t_{p1} in equation 6 and if it is not identical with RHS, t_{p2} is calculated as before by interpolation. This procedure is continued until the equation is satisfied. The t_{pn} equalising both sides of equation 6 is the required t_p that will maximise the yield at a given F. Generally a fair estimate of this t_p can be obtained by three or four interpolations.

Using the above method the optimum t_p has been calculated for $F = 0.73$ for the North Sea plaice. The required data taken from Beverton (MS, 1954) are $M = 0.1$, $Z = 0.83$, $t_p = 3.72$ years, $t_\lambda = 15$ years, $K = 0.095$, $t_0 = -0.815$ years, $W_\infty = 2867$ g and $\gamma = 3$. The optimum t_p for $F = 0.73$ is found to be 10.4219 years although from the equilibrium yield curve Beverton concludes that the optimum t_p is 10 years. From Table I it can be seen that

TABLE I. Yield per recruit $\frac{Y_w}{R}$ for plaice when F is 0.73 for values of t_p around the estimated best age of exploitation of 10.4219 years.

F = 0.73	
t_p years	$\frac{Y_w}{R}$ g
10.0	413.10
10.40	413.38
10.42	414.42
10.4219	414.42
10.43	414.44
10.44	414.40
10.50	414.39

10.4219 is a more accurate estimate of the optimum t_{γ} which will maximise the yield at $F = 0.73$.

Equation 6 can be used to solve for t_{γ} even when the growth exponent γ in length-weight relation is not three. When γ is three, the upper value in

the summation term $\sum_{n=0}^{d-1}$ will be three as also in equation 2 and the values of

the binomial coefficients γ_n will be successively 1, 3, 3, and 1. When r is not a positive integer, the accuracy of the optimum t_{γ} for a given F can be increased by increasing the number of terms in equation 6 although it is seldom necessary to take more than 8 or 10 terms. However, if a digital computer is available the total number of terms in equation 6 can be considerably increased.

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