

BAROTROPIC PLANETARY WAVES ON A RANDOM BOTTOM

Y.S. PRAHALAD*

C.P.I.R., Bombay University, Panaji, Goa, India

D. SENGUPTA

National Institute of Oceanography, Dona Paula, Goa, India

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The propagation of planetary waves on an irregular seabed is studied, with the roughness modelled as a stochastic process in one coordinate. Khasminskii's limit theorem is used to determine the probability density of the wave process. The mean wave is found to have an effectively changed wave number and to decay exponentially in the direction of phase propagation. The effect of the bottom roughness is strongly dependent on the wavelength.

1. Introduction

Planetary waves in the sea are rather sensitive to bottom topography. This is essentially so because the potential vorticity of a column of fluid changes as it cuts contours of f/H , f being the local Coriolis frequency and H the local depth.

Rhines and Bretherton [5] studied the effects of a sea floor with sinusoidal depth fluctuations in one direction on barotropic planetary waves. Thomson [6] investigated the effects of (one-dimensional) random topography on planetary wave propagation. He obtained the dispersion relation of the mean barotropic planetary wave using the method of smoothing, and he determined the deviation in phase speed and amplitude of the mean wave from those of the deterministic wave.

The present study concerns the propagation of barotropic planetary waves over a sea bed with small random inhomogeneities that are a function of one coordinate, superposed on a gentle exponential trend. As a generalization of this geometry, the case of the mean depth varying exponentially in two directions orthogonal to each other is also considered. In the latter configuration, the contours of mean depth are curved.

The depth fluctuations are modelled as a stochastic process in one coordinate. In Section 2, the stochastic ordinary differential equations (o.d.e.'s) governing the evolution of the amplitude and phase of the wave are set up for the two geometries described above. A well-known limit theorem due to Khasminskii is then used in Section 3 to model the wave process as a Markov process, and the expressions for the mean wave and the mean square amplitude are obtained. Section 4 is a discussion of the results, including estimates of their region of validity. The appendix contains the statement of Khasminskii's theorem.

* Current address: Department of Mathematics, Goa University, Bambulim, Goa 403005, India.

2. Stochastic o.d.e.'s for amplitude and phase evolution

On the β -plane with Cartesian axes x, y directed East and North, low frequency motion in a homogeneous sea of variable depth $H(x, y)$, with a rigid lid is governed by

$$\nabla_h^2 \left(\frac{\partial \Psi}{\partial t} \right) + \beta \frac{\partial \Psi}{\partial x} - \frac{1}{H} [f_0 \nabla_h \Psi \times \nabla_h H] \cdot \hat{k} = 0 \quad (2.1)$$

where ∇_h^2 is the two-dimensional Laplacian, $\Psi(x, y)$ is a transport stream function and \hat{k} denotes the unit vector directed vertically upwards; the Coriolis frequency f is written as $f = f_0 + \beta y$. Equation (2.1) has been obtained under the first and second approximations of Rhines (see [2]). For a sea shoaling exponentially to the North, $H(x, y)$ is taken in the form

$$\ln H(x, y) = \ln H_0 - y/l + \varepsilon \tilde{\mu}(y) \quad (2.2)$$

where l is a positive number and $\tilde{\mu}(y)$ is a stationary, zero-mean stochastic process in y , differentiable in the mean square sense. The small parameter ε is so chosen that the mean square value of $\tilde{\mu}(y)$ is of order unity. Ignoring the trend in the depth, the fractional depth change $\Delta = |H - H_0|/H_0$ is given in terms of ε by

$$\Delta = |\exp(\varepsilon \tilde{\mu}) - 1|. \quad (2.3)$$

If the flow develops a nonlinear regime at the sea bed, the wave will not feel the roughness of the bottom: A Rossby number R is therefore defined for the flow near the bottom as $R = V/f_0 L$, where V is a typical particle speed and L a length scale characteristic of the random depth fluctuations,¹ and it is required for linearity that $R/\Delta \ll 1$ [5]. Also, in order to satisfy the usual requirement that the fluid move as a column, the mean slope of the sea floor must be small.

On a smoothly sloping bottom, (2.1) admits plane wave solutions

$$\Psi = \exp\{i(k_1 x + k_2 y - \omega t)\}$$

with the dispersion relation

$$\omega = -k_1(\beta + f_0/l)/(k_1^2 + k_2^2). \quad (2.4)$$

Notice that for $l > 0$ the wave vector must have a westward component. The particle motions associated with these waves are along straight lines perpendicular to the wave vector.

Equations (2.1) and (2.4) give, with

$$\Psi(x, y, t) = \exp\{i(k_1 x - \omega t)\} \psi(y),$$

the Helmholtz equation

$$\frac{d^2 \psi}{dy^2} + k_2^2 \psi = -\varepsilon \frac{k_1 f_0}{\omega} \frac{d\tilde{\mu}(y)}{dy} \psi. \quad (2.5)$$

The solution of this equation is prescribed to be $\psi(y) = A(y) \cos[k_2 y + \theta(y)]$, where the amplitude and phase of the wave are random functions of y . $A(y)$ and $\theta(y)$ are assumed to be slowly varying with y so that

$$\frac{d\psi}{dy} = -A(y) k_2 \sin[k_2 y + \theta(y)], \quad (2.6)$$

¹ L may be taken to be the mean distance between successive zero-crossings of the randomly varying part of the depth.

and

$$\dot{\theta} = \frac{\dot{A} \cos \phi}{A \sin \phi}, \quad (2.7)$$

where $\phi \equiv k_2 y + \theta(y)$ and the overdot denotes differentiation with respect to y .

On using (2.5)-(2.7) the phase and the logarithm of the amplitude u are seen to be governed by

$$\dot{u} = \frac{1}{2}\varepsilon(k_1 f_0 / k_2 \omega) \dot{u} \sin 2\phi \quad \text{and} \quad \dot{\theta} = \frac{1}{2}\varepsilon(k_1 f_0 / k_2 \omega) \dot{u} (\cos 2\phi + 1). \quad (2.8)$$

The evolution of the wave thus depends on the process $\dot{u}(y)$, which is the derivative of the fluctuating part of the logarithm of the depth.

It is simple to extend the above to the geometry with curved depth contours: The Cartesian axes x, y make an angle α with East, North, and the depth is given by

$$\ln H = \ln H_0 - x/l_1 - y/l_2 + \varepsilon \tilde{u}(x). \quad (2.9)$$

This is the same configuration as in [6], and is the most general form of the sea bed for which the solution of the deterministic problem is known.

Under the same approximations as above, planetary wave motion is governed by the equation

$$\nabla_h^2 \Psi + \frac{i}{\omega} \left[(\beta \cos \alpha + f_0 / l_2) \frac{\partial \Psi}{\partial x} - (\beta \sin \alpha + f_0 / l_1) \frac{\partial \Psi}{\partial y} \right] = -\varepsilon \frac{if_0}{\omega} \left(\frac{d\tilde{\mu}}{dx} \right) \frac{\partial \Psi}{\partial y}, \quad (2.10)$$

the homogeneous part of which admits solutions

$$\Psi = \exp\{i(k_1 x + k_2 y)\},$$

with the dispersion relation

$$\omega = \frac{-k_1(\beta \cos \alpha + f_0 / l_2) - k_2(\beta \sin \alpha + f_0 / l_1)}{k_1^2 + k_2^2}. \quad (2.11)$$

With

$$\Psi(x, y) = e^{ik_2 y} \Psi(x) = e^{ik_2 y} \exp\{-i(\beta \cos \alpha + f_0 / l_2)x / 2\omega\} \psi(x),$$

eq. (2.10) reduces to

$$\frac{d^2 \psi}{dx^2} + \bar{k}_1^2 \psi = \varepsilon \frac{f_0 k_2}{\omega} \frac{d\tilde{\mu}}{dx} \psi \quad (2.12)$$

where

$$\bar{k}_1^2 = \left[\frac{1}{4\omega^2} (\beta \cos \alpha + f_0 / l_2)^2 - \frac{k_2}{\omega} (\beta \sin \alpha + f_0 / l_1) - k_2^2 \right].$$

With ψ written as $\psi = A(x) \cos[\bar{k}_1 x + \theta(x)]$, and under the assumption that A and θ vary slowly with x , the stochastic o.d.e.'s for the logarithm of the amplitude and the phase are

$$\dot{u} = -\frac{1}{2}\varepsilon \left(\frac{f_0 k_2}{\omega \bar{k}_1} \right) \dot{\mu} \sin 2\phi \quad \text{and} \quad \dot{\theta} = -\frac{1}{2}\varepsilon \left(\frac{f_0 k_2}{\omega \bar{k}_1} \right) \dot{\mu} (\cos 2\phi + 1) \quad (2.13)$$

The system (2.13) has the same form as (2.8), except for the negative sign and the different expression in the first bracket on the r.h.s.

Equations (2.8) and (2.13) are systems of coupled random differential equations whose solutions are random functions. The amplitude equations resemble the equations of motion of Brownian particles subject to a randomly fluctuating force, and one would expect these 'particles' to diffuse in the log-amplitude and phase space. A complete statistical description of these random functions (log-amplitude and phase) is given by the probability distribution.

The probability distribution of the wave process can be found if the slope process is stationary and has independent increments. The latter condition is highly restrictive; however, it can be shown that a weaker condition on the driving process enables us to determine the probability distribution of the solution process in a certain asymptotic sense. If the correlation function of the driving (slope) process falls off rapidly enough, then the process, appropriately scaled spatially, tends to a process with independent increments, and the scaled solution (wave) process converges to a diffusion process. This is the physical content of Khasminskii's theorem, which is invoked in the next section.

A variant of the theorem was used to study the transmission of wave energy through a slab of random medium by Kohler et al. [7]. One of the authors has applied the theorem to parametric amplification of Plasma waves [4] and to the acceleration of cosmic rays [3].

3. Probability density and moments of the wave process

The o.d.e.'s for both cases, (a) with straight depth contours, and (b) with curved depth contours, are of the form

$$\frac{dZ(t)}{dt} = \varepsilon F[Z(t), X(t), t],$$

and their r.h.s.'s satisfy the conditions for the applicability of Khasminskii's theorem (provided the process $\dot{\mu}$ is assumed to have a finite correlation length. The planetary wave process thus approximates a diffusion process with the Fokker-Planck equation given by

$$(a) \quad \frac{\partial P}{\partial y} = a_{1a} \frac{\partial P}{\partial u} + a_{2a} \frac{\partial P}{\partial \theta} + d_{1a} \frac{\partial^2 P}{\partial u^2} + d_{2a} \frac{\partial^2 P}{\partial \theta^2} \quad (3.1)$$

and

$$(b) \quad \frac{\partial P}{\partial x} = a_{1b} \frac{\partial P}{\partial u} + a_{2b} \frac{\partial P}{\partial \theta} + d_{1b} \frac{\partial^2 P}{\partial u^2} + d_{2b} \frac{\partial^2 P}{\partial \theta^2}. \quad (3.2)$$

The drift and diffusion coefficients (see Appendix) are

$$a_{1a} = 2d_{1a} = 2d_{2a} = \varepsilon^2 C_a^2 k_2^2 \operatorname{Re}\{s(2k_2)\} \quad \text{and} \quad a_{2a} = \varepsilon^2 C_a^2 \left[\frac{1}{2} k_2 R(0) - k_2^2 \operatorname{Im}\{s(2k_2)\} \right], \quad (3.3)$$

where $\operatorname{Re}\{\cdot\}$ and $\operatorname{Im}\{\cdot\}$ denote real and imaginary parts,

$$R(0) = \langle \tilde{\mu}^2 \rangle, \quad s(p) = \int_0^\infty \langle \tilde{\mu}(y) \tilde{\mu}(y+z) \rangle e^{ipz} dz.$$

The angular brackets denote expectation, and $C_a = (k_1 f_0 / k_2 \omega)$; The expressions for the coefficients a_{1b} , a_{2b} , d_{1b} and d_{2b} are identical except that C_a is replaced by C_b , and k_2 by \bar{k}_1 , where $C_b = (k_2 f_0 / \bar{k}_1 \omega)$ and, of course,

$$s(p) = \int_0^\infty \langle \tilde{\mu}(x) \tilde{\mu}(x+z) \rangle e^{ipz} dz.$$

Case (a): With the initial condition on the transition probability density

$$P(u, \theta, y | u=0, \theta=0, y=0) = \delta(u)\delta(\theta),^2$$

eq. (3.1) gives the probability density of the u, θ process as

$$P(u, \theta, y) = (4\pi d_{1a}y)^{-1/2} \exp\left\{\frac{-(u + a_{1a}y)^2}{4d_{1a}y}\right\} (4\pi d_{2a}y)^{-1/2} \exp\left\{\frac{-(\theta + a_{2a}y)^2}{4d_{2a}y}\right\}. \quad (3.4)$$

Using $P(u, \theta, y)$, it is simple to find any averaged quantity pertaining to the wave field. The mean square amplitude is

$$\langle A^2 \rangle = \langle e^{2u} \rangle = \exp\{2(2d_{1a} - a_{1a})y\} = 1. \quad (3.5)$$

The mean wave may be written as the product of the mean amplitude and the mean phase factor, since u and θ are independently distributed:

$$\begin{aligned} \langle \psi \rangle &= \langle A \rangle \langle \cos(k_2 y + \theta) \rangle = \exp\{(d_{1a} - a_{1a} - d_{2a})y\} \cos[(k_2 - a_{2a})y] \\ &= \exp\{-[\varepsilon^2 k_2^2 C_a^2 \operatorname{Re}\{s(2k_2)\}]y\} \cos\{[k_2 + \varepsilon^2 k_2^2 C_a^2 (\operatorname{Im}\{s(2k_2)\} - R(0)/2k_2)]y\}. \end{aligned} \quad (3.6)$$

From (2.5) and (2.6), it can be seen that the mean wave $\langle \Psi \rangle$ has an effectively changed y -wavenumber $k_{2\text{eff}}$ given by

$$k_{2\text{eff}} = k_2 [1 + \varepsilon^2 k_2^2 C_a^2 (\operatorname{Im}\{s(2k_2)\} - R(0)/2k_2)], \quad (3.7)$$

and decays exponentially along y with a decay length l_d given by

$$l_d = [\varepsilon^2 k_2^2 C_a^2 \operatorname{Re}\{s(2k_2)\}]^{-1}. \quad (3.8)$$

Using the expression for C_a , and the dispersion relation (2.4), l_d may be written as

$$l_d = \frac{(\beta + f_0/l)^2 \lambda^4}{(2\pi)^4 \varepsilon^2 f_0^2 \operatorname{Re}\{s(2k_2)\}}, \quad (3.9)$$

where λ is the wavelength of the planetary wave Ψ , $\lambda = 2\pi/(k_1^2 + k_2^2)^{1/2}$. Thus the shorter the wave, the more strongly is it damped by the bottom randomness.

Case (b): With the initial condition

$$P(u, \theta, x | u=0, \theta=0, x=0) = \delta(u)\delta(\theta),$$

the solution of (3.2),

$$P(u, \theta, x) = (4\pi d_{1b}x)^{-1/2} \exp\left\{\frac{-(u + a_{1b}x)^2}{4d_{1b}x}\right\} (4\pi d_{2b}x)^{-1/2} \exp\left\{\frac{-(\theta + a_{2b}x)^2}{4d_{2b}x}\right\},$$

permits the calculation of $\langle A^2 \rangle$ and $\langle \psi \rangle$. The results are $\langle A^2 \rangle = 1$, as before, and

$$\langle \psi \rangle = \exp\{-[\varepsilon^2 \bar{k}_1^2 C_b^2 \operatorname{Re}\{s(2\bar{k}_1)\}]x\} \cos\{\bar{k}_1 [1 + \varepsilon^2 \bar{k}_1 C_b^2 (\operatorname{Im}\{s(2k_2)\} - R(0)/2\bar{k}_1)]x\}. \quad (3.10)$$

In this case the planetary wave Ψ has a wavevector $[k_1, k_2]$ with

$$k_1 = \bar{k}_1 - \frac{1}{\omega} (\beta \cos \alpha + f_0/l_2)$$

² The condition signifies that the wave is deterministic at $y=0$.

(see eq. (2.12)). However, the extra factor in $\langle \Psi \rangle$, namely

$$\exp\left\{-\frac{1}{2\omega}(\beta \cos \alpha + f_0/l_2)x\right\},$$

leaves the decay term in (3.10) unaffected, and the mean planetary wave $\langle \Psi \rangle$ therefore has a decay length l_d given by

$$l_d = [\varepsilon^2 \bar{k}_1^2 C_b^2 \operatorname{Re}\{s(2\bar{k}_1)\}]^{-1} = \frac{\lambda^4 [(\beta \cos \alpha + f_0/l_2) \cot \Phi + (\beta \sin \alpha + f_0/l_1)]^2}{(2\pi)^4 \varepsilon^2 f_0^2 \operatorname{Re}\{s(2\bar{k}_1)\}} \quad (3.11)$$

where, as in (a) before,

$$\lambda = 2\pi/(k_1^2 + k_2^2)^{1/2} \quad \text{and} \quad [k_1, k_2] = \left[\frac{2\pi}{\lambda} \cos \Phi, \frac{2\pi}{\lambda} \sin \Phi \right],$$

i.e. the wave vector makes an angle Φ with the x axis. When $\Phi = 0$ or π , (3.11) shows that the wave suffers no attenuation. This is to be expected, since the particle motion for $k_2 = 0$ is entirely along the ridges and valleys of the roughness, and not across the grain. (This phenomenon is not observed in case (a): There a planetary wave with wavenumber purely in the y -direction does not exist.)

4. Discussion

It would appear that there exist large regions in the world ocean where the exponential trend(s) with a superposed one-dimensional random roughness constitutes a reasonably good description of the topography (e.g. [6]). However, quite apart from the question of whether one-dimensional random fluctuations of depth are realised in the sea, there is the issue of the region of validity of the present treatment of planetary waves in such a geometry.

For specificity, consider case (a). Let $\tilde{\mu}(y)$ be a wide-band process with its spectral density $s(p)$ significant up to an upper cut-off at $p \sim O(0.1) \text{ km}^{-1}$. Let l be equal to a few earth radii, and $\varepsilon = 0.025$ which corresponds to $\Delta \approx 0.025$ i.e. 100 m ups and downs in a sea 4 km deep. While it is not claimed that these conditions are representative of the sea floor everywhere, it is conceivable that they could be found in the ocean.

Since

$$\langle \tilde{\mu}^2 \rangle \sim O(1), \operatorname{Re}\{s(p)\} \sim O(10) \text{ km}$$

if it is uniform over the range of p . A wave with y -wavenumber k_2 will not be affected by the bottom randomness if $s(2k_2)$ is zero.

With the above values of the relevant quantities, the decay length of the mean planetary wave,

$$l_d = \frac{(\beta + f_0/l)^2 \lambda^4}{(2\pi)^4 \varepsilon^2 f_0^2 \operatorname{Re}\{s(2k_2)\}}$$

is

$$l_d \approx 1.6 \times 10^{-8} \lambda^4 \text{ km}. \quad (4.1)$$

Thus for $\lambda = 500 \text{ km}$, $l_d \approx 500 \text{ km}$ and for smaller wavelengths, l_d is smaller than the wavelength itself.

It is proper to claim that the ansatz of slow variation of the wave (eq. (2.6)) and, consequently, the application of Khasminskii's theorem, is justified only if the scale over which the mean wave varies is large compared to a wavelength. Therefore, with the above values of l , ε and $\text{Re}\{s(2k_2)\}$, the results of the previous sections are taken to be valid only for λ greater than 850 km or so, for which l_0 is more than ten times λ . For smaller Δ , the lower limit on λ can be suitably relaxed.

Rhines and Bretherton's [5] analysis indicates that sinusoidal roughness of the sea bed has rather severe effects on the propagation of planetary waves. This would also seem to be the case for relatively small-scale waves on a randomly rough bottom. However, the influence of the bottom on sufficiently long waves is milder, permitting meaningful results to be obtained for the latter.

For a wave of given x - and y -wavenumbers, the energy of the planetary wave field per unit area is proportional to A^2 . Therefore, the result $\langle A^2 \rangle = 1$ implies that for all y , the energy resides entirely in the monochromatic wave $e^{ik_1x} \cos k_2y$, and perhaps in its backscattered counterpart with k_2 negative. This question will be taken up in a later article, but it may be pointed out here that the occurrence of $s(2k_2)$ in the analysis indicates that the bottom roughness acts to reverse the y -component of the wave vector (i.e. gives rise to backscatter).

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Appendix. Khasminskii's theorem

Khasminskii's [1] theorem is the analogue, for stochastic processes, of the classical Central Limit theorem. It shows that under very general conditions on the r.h.s.'s of eqs. (2.8) and (2.13), the wave processes become Markovian and obey a diffusion equation for the transition probability density $P(u, \theta | u_0, \theta_0)$.

Consider the stochastic o.d.e.

$$\frac{dZ(t)}{dt} = \varepsilon F[Z(t), X(t), t], \quad (\text{A.1})$$

where ε is a parameter that is small in the sense of perturbation theory, $Z(t)$ and $X(t)$ are stochastic processes taking values in an Euclidean space \mathbb{R}^n , and F is a measurable function of $Z(t)$, $X(t)$ and t , with values in \mathbb{R}^n .

Let the following be true:

- (1) $\langle F[Z(t), X(t), t] \rangle = 0$
- (2) There exists some constant c such that
 - (a) $|F_i| < c$,
 - (b) $|\partial F_i / \partial Z_j| < c$
 - (c) $|\partial^2 F_i / \partial Z_j \partial Z_k| < c$, $i, j, k = 1, 2, \dots, n$

(where Z_i and F_i are the components of Z and F , uniformly in Z , X and t .)

(3) The limits

$$(a) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^s \left\langle \sum_{j=1}^n \frac{\partial F_i}{\partial Z_j} [Z, X(s), s] F_j [Z, X(\sigma), \sigma] \right\rangle d\sigma ds = a'_i(Z)$$

and

$$(b) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^s \langle F_i [Z, X(s), s] F_j [Z, X(\sigma), \sigma] \rangle d\sigma ds = d'_{ij}(Z) \quad (A.2)$$

for fixed Z and $T \geq 0$ exist uniformly in t_0 and Z .

(4) The stochastic process $X(\sigma)$ has the following mixing property:

$$\sup_{\substack{A \in U'_\delta \\ B \in U^+_\delta}} |P(B/A) - P(B)| < \rho(s), \quad \rho(s) \sim O\left(\frac{1}{s^{\epsilon+\delta}}\right), \quad \delta > 0$$

where $U'_\delta, 0 \leq s \leq t$ denotes the σ -algebra of events generated by $X(\sigma), s \leq \sigma \leq t$, A and B are arbitrary sets, $P(A)$ is the probability of event A and $P(B|A)$ is the conditional probability of B given A .

Now let $\tau = \epsilon^2 t$, $Z^\epsilon(\tau) = Z(\tau/\epsilon^2)$. Khasminskii's theorem says that under the above hypotheses, $Z^\epsilon(\tau)$ converges weakly as $\epsilon \rightarrow 0$ and $t \rightarrow \infty$ with $\epsilon^2 t$ fixed, to the Markov process $Z^0(\tau)$ whose generator is

$$L_0 = \sum_{i=1}^n a'_i(Z) \frac{\partial}{\partial Z_i} + \sum_{i,j=1}^n d'_{ij}(Z) \frac{\partial^2}{\partial Z_i \partial Z_j}. \quad (A.3)$$

Condition (4) above just requires $X(\sigma_1)$ and $X(\sigma_2)$ to become independent random variables sufficiently rapidly as $|\sigma_1 - \sigma_2| \rightarrow \infty$.

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