

NUMERICAL CALCULATION OF WAVE REFRACTION

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ABSTRACT

The numerical method employed for the solution of wave refraction equations is discussed. The computer program for the computation of the coordinates of the wave rays and the refraction and shoaling coefficients along the wave ray is presented as appendix.

1. INTRODUCTION

The wind waves in the near shore regions account for major portion of the environmental forces and significantly influence the planning, design and construction of shore protection works, harbours, waterways and coastal structures. When the waves from deep water move into the shallow water of the nearshore region, they undergo refraction due to interaction with the bottom topography. Wave refraction leads to local increase or decrease in wave energy and consequently wave heights and wave forces. The other wave topography interaction effects such as wave reflection from the sloping bottom and loss of energy due to friction at the bottom are generally small and are neglected. Hence, for the planning of any developmental activity in the nearshore regions the wave refraction pattern and the details of the wave forces in that region become a necessary pre-requisite. The collection of field data on wave topography interaction is expensive and for long, this problem was studied through the construction of wave refraction pattern using graphical procedures. In recent years, with the availability of fast computers, the numerical method is found to be more convenient for the construction of wave refraction pattern. This paper describes the numerical method employed for the solution of wave refraction equations and also presents the computer program developed for the computation of wave parameters along a wave ray.

2. WAVE REFRACTION EQUATIONS

The computational algorithm used in the program presented here is based on the method described by Arthur, Munk and Isaacs(1952) for the construction of wave refraction pattern. The basic assumption underlying the wave refraction equations is that the linear small amplitude wave theory is applicable in the near shore regions. From this theory we know that the wave speed depends on the depth of water in which it propagates. Hence, when a wave moves into the shallow region

with nonuniform depths, the wave speed along the wave crest varies and the part of the wave crest in deeper water moves faster than the part in shallower water. This variation causes the wave crest to bend and consequently changes the wave propagation direction along the crest. The lines drawn perpendicular to the wave crest are called the wave rays or wave orthogonals. Fig. 1 shows adjacent wave orthogonals O_1 and O_2 and wave fronts F_1 and F_2 .

The assumptions generally made in the computation of wave refraction problems are

- (a) Wave energy transmitted between adjacent wave orthogonals remains constant,
- (b) the direction of wave advance is perpendicular to the wave crest,
- (c) waves have small amplitude, constant period and long crests,
- (d) speed of a wave of given period at any location depends only on the depth of water at that location,
- (e) changes in bottom topography are gradual, and
- (f) effects of currents and winds are considered negligible.

It is readily seen from the first assumption that, between the same orthogonals

$$bh^2 C_g = b_o h_o^2 C_{g_o}, \quad (1)$$

Since energy at any location is proportional to the distance 'b' between the wave orthogonals, the square of wave height h and the group velocity C_g . The subscript 'o' refer to parameters in deep water. Then from equation (1) we have

$$h = h_o K_1 K_2 \quad (2)$$

where $K_1 = \sqrt{b_o/b}$ is called the refraction coefficient,

$K_2 = \sqrt{C_{g_o}/C_g}$ is called the shoaling coefficient,

$$C_g = \frac{C}{2} + \frac{2\pi d}{T \sin h (4\pi d/CT)}$$

C is the wave speed, T is the period of the wave, d is the depth of water and $C_{g_o} = C/2$.

The basic equations describing the wave orthogonals are (See Ref. 2 for detailed derivations),

$$\frac{dx}{ds} = \cos \theta \quad (3)$$

$$\frac{dy}{ds} = \sin \theta \quad (4)$$

$$\frac{d\theta}{ds} = \frac{1}{c} \left(\sin \theta \frac{\partial c}{\partial x} - \cos \theta \frac{\partial c}{\partial y} \right) \quad (5)$$

$$\text{and } \frac{d^2\beta}{ds^2} + P(s) \frac{d\beta}{ds} + q(s)\beta = 0, \quad (6)$$

$$P(s) = -\frac{1}{c} \left(\cos \theta \frac{\partial c}{\partial x} + \sin \theta \frac{\partial c}{\partial y} \right),$$

$$q(s) = \frac{1}{c} \left(\sin^2 \theta \frac{\partial^2 c}{\partial x^2} - \sin 2\theta \frac{\partial^2 c}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 c}{\partial y^2} \right),$$

Where 's' is the distance along a wave ray measured from a desired point in deep water (see fig. 1), x and y are the cartesian coordinates of the chosen wave ray, θ is the angle between the tangent to the wave ray and the x-axis, c is the wave speed at any location and β is the distance between adjacent orthogonals relative to their separation in deep water. Now, the solution to equations (3) to (5) with proper initial conditions gives the cartesian coordinates of the wave ray, and the solution to equation (6) gives the variation of β , i.e. the refraction coefficient ($K_1 = \sqrt{\frac{1}{\beta}}$) along the wave ray.

3. DETAILS OF NUMERICAL COMPUTATION

The problem of obtaining the wave refraction pattern for the given area reduces now to the computation of coordinates of wave orthogonals and the refraction coefficients, $K_1 = \sqrt{1/\beta}$, which can be obtained by solving the set of differential equations (3) to (6). Runge-Kutta-Gill (see M.J. Romanelli, 1960) integration procedure is used here for solving the above differential equations.

The area of study is divided into a rectilinear grid system as shown in figure 2, with equal grid spacing in both directions. The grid spacing is chosen as the unit of measurement for lengths in horizontal plane. This choice does not affect the

unit for wave speed since the wave speed and its derivatives always appear as a ratio in the equations (5) and (6). The x-axis of the grid system is chosen in the deep water, which is defined as the water having a minimum depth equal to half the longest wave length of the wave spectrum being studied. Grid spacing will primarily be determined by the complexity of the bottom topography and must be small enough to obtain the required accuracy.

The water depths H_{ij} at each grid point forms the basic data needed for the computation of wave speed C_{ij} . The other input data needed are the deep water wave characteristics, viz. the wave period and the direction in deep water. Now for a given wave period, the wave speed C_{ij} can be calculated using the relation

$$C_{ij} = \frac{gT}{2\pi} \tanh \left(\frac{2\pi H_{ij}}{T} \right)$$

by an iterative procedure at all grid points, starting from the deep water end of the study area; the deep water wave speed provides the initial approximation for the iterative procedure. For computing the wave speed at subsequent grid points, the value calculated at the previous grid point serves as the initial approximation. For depths less than $L/20$, where L is the deep water wave length, the wave speed was computed from $C_{ij} = \sqrt{g H_{ij}}$. Whenever H_{ij} is less than 0.1 meter, C_{ij} was assumed as zero.

It is necessary next to calculate the partial derivatives of the wave speed with respect to x and y at all grid points. With grid spacing as the unit of measurement for lengths in the horizontal plane, the finite difference form of the differential coefficients are,

$$\frac{\partial C}{\partial x_{i,j}} = (C_{i+1,j} - C_{i-1,j}) / 2$$

$$\frac{\partial C}{\partial y_{i,j}} = (C_{i,j+1} - C_{i,j-1}) / 2$$

$$\frac{\partial^2 C}{\partial x_{i,j}^2} = C_{i+1,j} - 2C_{i,j} + C_{i-1,j}$$

$$\frac{\partial^2 C}{\partial y_{i,j}^2} = C_{i,j+1} - 2C_{i,j} + C_{i,j-1}$$

$$\frac{\partial^2 c}{\partial x \partial y^2}_{i,j} = (c_{i+1, j+1} - c_{i-1, j+1} - c_{i+1, j-1} + c_{i-1, j-1})/4$$

The wave speed and its derivatives at any point inside the grid may be obtained by linear interpolation. For example, to calculate c at P , we first calculate c at Q and R by linear interpolation (see Figure 3). Another linear interpolation between Q and R will give the required value at P .

INITIAL CONDITIONS

The integration of the differential equations (3) to (6) is started offshore in the deepwater where the wave rays are parallel, and proceeds towards the shore. The integration step size is expressed as a fraction of the spacial grid spacing, the unit of measurement for lengths in horizontal plane. As the x axis of the coordinate system is in deep water, the origin for the wave rays may be selected at equal intervals along the x -axis where the refraction coefficient is 1, i.e. $\beta = 1$, and its derivative is zero and θ will be equal to the deep water wave direction. These conditions along the x -axis form the initial conditions for the differential equations (3) to (6).

4. DETAILS OF THE COMPUTER PROGRAM

DATA INPUT

The data input required to trace the orthogonals and to calculate the refraction coefficients are:

- NX - number of grid points along the x -axis
- NY - number of grid points along the y -axis
- NPRNT - if zero the depths at grid points will not be printed
- T - wave period
- THE - direction of the wave with respect to x -axis in deep water
- X1, Y1 - is the origin of the first orthogonal.
The grid system is chosen such that the water depths along the x -axis are greater than half the longest wave length of the wave spectrum being studied. The starting points for the orthogonals are chosen along the x -axis at regular intervals of 'XST'. The starting point of the first orthogonal should be beyond the second grid point along the x -axis and that of the

last orthogonal should be before (NX-1)th grid point, since the derivatives of wave speeds are not available along the first and the last verticals of the grid. The integration will be performed between the second and (NX-1)th vertical where the wave speed and its derivatives are available and hence, it is the area where the integration of the given equation are possible.

- NSET - is the number of orthogonals traced
- STEP - is the integration step size expressed as a fraction of the spatial grid spacing.
- LAST - maximum possible number of steps involved in the integration.
- H(I,J) - the water depths are read into this array. Wave speed is computed from the known depths H(I,J) and stored in the array D(I,J).

Detailed program listing is given in Appendix-A.

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APPENDIX A

PROGRAM LISTING

PROGRAM WAVE REFRACTION

DEPTHS AT GRID POINTS ARE READ INTO THE MATRIX H(I,J).

C DIMENSIONS OF H(N,M) AND D(N,M) ARE SAME. N.&E.NX AND
 1M.GE.NY.
 C DIMENSIONS OF OTHER ARRAYS ARE FIXED.
 C IF NPRNT=0, INPUT DATA MATRIX WILL NOT BE PRINTED.
 C SUBROUTINE DATA HAS TO BE WRITTEN BY THE USER AS PER HIS
 REQUIREMENT.

```

DIMENSION H (25,30), D(25,30)
DIMENSION X(10), Z(10)
READ (7,100) NX,NY, NPRNT
READ (7,103) NSET,X1,Y1,XST
READ (7,103) LAST,STEP
READ (7,102) T,THE
CALL DATA (NX,NY,H)
100 FORMAT (5I5)
102 FORMAT (2F4.0)
103 FORMAT (I5,5F5,1)
C=9.80665
PI=3.141593
EP=0.001
X1=X1+1.
Y1=Y1+1.
IF(NPRNT.EQ.0) GO TO 21
WRITE (8,301)
DO 20 J=1,NY
WRITE (8,300)J

20 WRITE (8,205) (H(I,J), I=1,NX)
21 CONTINUE
CALCULATE C(I,J)
CA=G*1/2./PI
DO 12 I=1,NX
CC=CA
DO 12 J=1,NY
DD=H(I,J)
12 D(I,J)=SPEED (T,PI,G,CA,DD,CC)
DO 15 ISET=1,NSET
NEQ=6
XEND=NX-EP
YEND=NY-EP
DO 16 I=1,NEQ
Z(I)=0.
16 X(I)=0.
C INITIAL CONDITION
X(2)=X1+ISET*XST-XST
X(3)=Y1
X(4)=THE*PI/180
X(5)=1
X(6)=0.
XP=X(2)-1.
YP=Y1-1.
I=X(2)+EP

```

```

J=Y1+EP
HP=H(I,J)
RC=SQRT(1./X(5))
WRITE (8,206)T,THE,ISET,XP,YP
JJ=0
SC=1.
WRITE (8,207)JJ,X(1),XP,YP,HP,CA,THE,X(5),RC,SC

```

```

15 CONTINUE
205 FORMAT (1H+, (10X,10(3X,F7.2)))
206 FORMAT (1H1, //5X, 13HWAVE PERIOD=F6.1, 4H SEC, 5X, 15HWAVE
1DIRECTION=F6.1//.5X, 5H ISET= I5, 5X, 2HX=F5.1, 5X, 2HY=F5.1//)
207 FORMAT (6X, 2HNO, 4X, HS, 6X, HX, 6X, HY, 4X, 5HDEPTH, 4X, HC, 4X,
15H I HETA, 3X, 4H BETA, 8X, 2HK1, 6X, 2HK2//5X, I3, 6(X, F6.2), E11.4, 2
2(X, F7.3))
300 FORMAT (/5X, I5)
301 FORMAT (1H1, //5X, 21H DEPTHS AT GRID POINTS /)
STOP
END

```

```

SUBROUTINE DATA (NX,NY,H)
DIMENSION H(25,30)

```

C DEPTHS AT GRID POINTS ARE READ INTO THE MATRIX H(I,J)
C SUBROUTINE DATA HAS TO BE WRITTEN BY THE USER AS PER
HIS REQUIREMENT.

```

READ (7,104) (H(1,J),J=1,NY)
104 FORMAT (23F3.1)
DO 25 J=1,NY
DO 25 I=2,NX
25 H(I,J)=H(1,J)
RETURN
END

```

```

SUBROUTINE RUNGA (NEQ,X,Z,H,D, LAST,STEP,XEND,YEND,T,PI,G,CA)
DIMENSION H(25,30), D(25,30)
DIMENSION X(10),Y(10),Z(10)
DIMENSION A(4),B(4),C(4)

```

```

DATA A/.5, .29289322, 1.7071068, .16666667/
DATA B/2., 1., 1., 2./
DATA C/.5, .29289322, 1.7071068, .5/
Y(1)=1.

```

```

DO 15 NST=1, LAST
ST=STEP

```

```

DO 12 ITER=1, 4
IF(X(2).LT.1.999) GO TO 16
IF(X(3).LT.0.999) GO TO 16
IF(X(2).GE.XEND) GO TO 16
IF(X(3).GE.YEND) GO TO 16
IF(NST.GT.1.AND.ITER.EQ.1) GO TO 20
CALL INTPOL(X(2),X(3),H,D,HL,CL,XL,YL,XXL,YYL,XYL,T,PI,G,CA)

```

```

20 CONTINUE

```



```

IF (CL.EQ.0) RETURN
X4=X(4)
Y(2)=COS(X4)
Y(3)=SIN(X4)
Y(4)=(SIN(X4)*XL-COS(X4)*YL)/CL
Y(5)=X(6)
PS=(COS(X4)*XL+SIN(X4)*YL)/CL
QS=(SIN(X4)**2*XXL-2*SIN(X4)*COS(X4)*XYL+COS(X4)**2*YYL)/CL
Y(6)=PS*X(6)-QS*X(5)
DO 12 J=1,NEQ
HK=(Y(J)-B(ITER)*Z(J))*A(ITER)
Z(J)=Z(J)+3*HK-C(ITER)*Y(J)
X(J)=X(J)+HK*ST
12 CONTINUE
TH=X(4)*180./3.141593
XP=X(2)-1.
YP=X(3)-1.
CALL INTPOL(X(2),X(3),H,D,HL,CL,XL,YL,XXL,YYL,XYL,T,PI,G,CA)
IF(X(5).LE.1.E-2) GO TO 13
RC=SQRT(1./X(5))
13 IF(CL.EQ.0.) GO TO 17
EX=EXP(4*PI/CL/T*HL)
SC=SQRT(CA/CL/(1.+8.*PI*HL/(CL*T*(EX-1./EX))))
IF(X(5).LE.1.E-2) GO TO 14
WRITE(8,200)NST,X(1),XP,YP,HL,CL,TH,X(5),RC,SC
GO TO 15
14 WRITE(8,201)NST,X(1),XP,YP,HL,CL,TH,X(5),SC
15 CONTINUE
RETURN
17 WRITE(8,200)NST,X(1),XP,YP,HL,CL,TH,X(5)
16 CONTINUE
200 FORMAT(5X,13,6(X,F6.2),E11.4,2(X,F7.3))
201 FORMAT(5X,I3,6(X,F6.2),E11.4,4X,2H??,3X,F7.3)
RETURN
END

```

```

SUBROUTINE INTPOL(X,Y,H,D,HL,CL,XL,YL,XXL,YYL,XYL,T,PI,G,CA)
DIMENSION H(25,30),D(25,30)
EP=.001
ISET=0
I=X+EP
J=Y+EP
HA=H(I,J)
HB=H(I+1,J)
HC=H(I,J+1)
HD=H(I+1,J+1)
GO TO 90
10 HL=AL
CC=D(I,J)
IF(CC.EQ.0.)CC=SQRT(HL*G)
CL=SPEED(T,PI,G,CA,HL,CC)
HA=0.

```

```

HB=0.
IF(J.EQ.1) GO TO 14
HA=(D(I+1,J)-D(I-1,J))/2.
HB=(D(I+2,J)-D(I,J))/2.
14 HC=(D(I+1,J+1)-D(I-1,J+1))/2.
HD=(D(I+2,J+1)-D(I,J+1))/2.
GO TO 90
16 XL=AL
HA=(D(I,2)-D(I,1))/2.
HB=(D(I+1,2)-D(I+1,1))/2.
IF(J.EQ.1) GO TO 18
HA=(D(I,J+1)-D(I,J-1))/2.
HB=(D(I+1,J+1)-D(I+1,J-1))/2.
18 HC=(D(I,J+2)-D(I,J))/2.
HD=(D(I+1,J+2)-D(I+1,J))/2.
GO TO 90
20 YL=AL
HA=0.
HB=0.
IF(J.EQ.1) GO TO 22
HA=D(I+1,J)-2.*D(I,J)+D(I-1,J)
HB=D(I+2,J)-2.*D(I+1,J)+D(I,J)
22 HC=D(I+1,J+1)-2.*D(I,J+1)+D(I-1,J+1)
HD=D(I+2,J+1)-2.*D(I+1,J+1)+D(I,J+1)
GO TO 90
24 XKL=AL
HA=D(I,2)-D(I,1)
HB=D(I+1,2)-D(I+1,1)
IF(J.EQ.1) GO TO 26
HA=D(I,J+1)-2.*D(I,J)+D(I,J-1)
HB=D(I+1,J+1)-2.*D(I+1,J)+D(I+1,J-1)
26 HC=D(I,J+2)-2.*D(I,J+1)+D(I,J)
HD=D(I+1,J+2)-2.*D(I+1,J+1)+D(I+1,J)
GO TO 90
28 YYL=AL
HA=(D(I+1,2)-D(I-1,2))/4.
HB=(D(I+2,2)-D(I,2))/4.
IF(J.EQ.1) GO TO 30
HA=(D(I+1,J+1)+D(I-1,J-1)-D(I-1,J+1)-D(I+1,J-1))/4.
HB=(D(I+2,J+1)+D(I,J-1)-D(I,J+1)-D(I+2,J-1))/4.
30 HC=(D(I+1,J+2)+D(I-1,J)-D(I-1,J+2)-D(I+1,J))/4.
HD=(D(I+2,J+2)+D(I,J)-D(I,J+2)-D(I+2,J))/4.
90 ISET=ISET+1
AA=HA+(HB-HA)*(X-I)
BB=HC+(HD-HC)*(X-I)
AL=AA+(BB-AA)*(Y-J)
GO TO (10,16,20,24,28,32),ISET
32 XYL=AL
RETURN
END

```

```
FUNCTION SPEED(T,PI,G,CA,DD,CC)
  NTIME=40
  EP=.01
  CN=0.
  IF(DD.LT..1) GO TO 12
  IF(DD.LT.CA*T/20.))CC=SQRT(G*DD)
  IF(DD.LT.CA*T/22.) GO TO 11
  DO 10 K=1,NTIME
  EX=EXP(2*PI*DD/CC/T)
  CN=CA*(EX-1./EX)/(EX+1./EX)
  CB=ABS(CC-CN)
  IF(CB.LT.EP) GO TO 12
10  CC=CN
    WRITE(8,200)K,CB
11  CN=CC
12  SPEED=CN
200  FORMAT( 5X, 15, 16HABSOLUTE(CC-CN) = E15.5,5X,
119HIN SUBROUTINE SPEED)
    RETURN
    END
```

.....

FIG. 3. GRID DESCRIPTION FOR INTERPOLATION

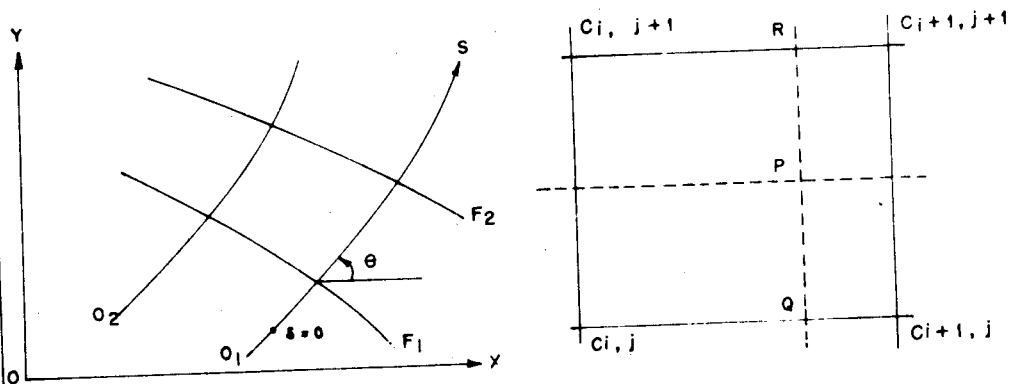


FIG. 1. ADJACENT WAVE ORTHOGONALS AND WAVE FRONTS.

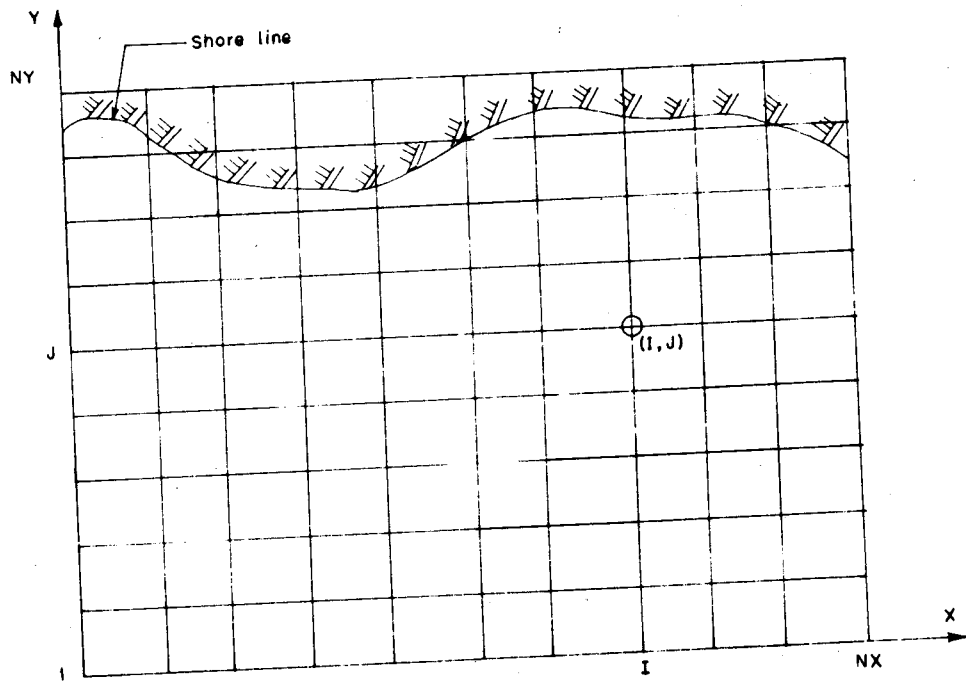


FIG. 2 GRID DESCRIPTION OVER THE STUDY AREA