

TRANSFORMATION OF COORDINATES FOR STORM SURGE SIMULATION

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ABSTRACT

The algorithm of Wanstrath et al (1976) for generating the transformation function which conformally maps the interior region bounded by the shore line and the sea-ward boundary to a rectangular region is discussed. This transformation, minimizes the difficulties associated with the irregular boundaries and improves the accuracy of the estimates in surge simulation problems. Using this algorithm the mapping function is obtained for specified regions along the east coast of India.

1. INTRODUCTION

The storm surge defined as the response of the sea level near the coast to the atmospheric force field of an approaching storm is generally simulated by solving the vertically integrated form of the long wave equations. The general difficulties of this simulation problem are the representation of the irregular shore line and the application of the shore boundary conditions. In most of the studies connected with the numerical simulation of long wave equations the irregular boundaries are approximated by orthogonal straight line segments, called the stair-step representation. Such representations not only involve complexities in specifying the boundary conditions along the shore at the computational stage but also, are likely to introduce inaccuracy in surge estimates near the shore, which precisely is the region of interest in surge simulation problems.

In order to minimize the difficulties associated with the irregular boundaries, Wanstrath et al (1976) first transformed the specified region into a rectangular region through a conformal mapping before obtaining the solution to the governing equations in the transformed plane. Thus the application of this procedure to surge simulation involves two stages of computation; the first stage is to find the transformation function which conformally maps the interior region bounded by the shoreline and the sea-ward boundary to a rectangular region and the next stage is to solve the transformed differential equation in the transformed rectangular region - the transformation being orthogonal, the modification to the governing equations are minimum.

This study is restricted to the development of a computer program to generate the transformation function to conformally map

the interior region bounded by an arbitrary curve into the interior region of a rectangle. The computer program developed is based on that of Wanstrath (1976) and it also incorporates certain improvements. Using this program the transformation coefficients for two overlapping regions along the east coast of India that have experienced severe storm surges, were computed.

It may be pointed out here that this transformation technique is useful in simulation of certain processes in coastal areas which require the consideration of more realistic boundaries; for example, Reid and Vastano (1965) have used a similar technique for solving the problem of refraction of long gravity waves near islands.

2. CONFORMAL MAPPING

In this section the algorithm of Wanstrath *et al* (1976) for generating the transformation function to transform any specified region to a rectangular region is discussed. Figure 1 shows the region R in Z -plane which is to be transformed into the interior of a rectangle in ζ -plane. The region R is bounded by the coastline, the sea-ward boundary and parallel lateral boundaries. The sea-ward boundary is taken at a place where the depth of the sea bed increases rapidly, i.e. at the shelf break. The 200-meter depth contour is generally assumed to represent the shelf break.

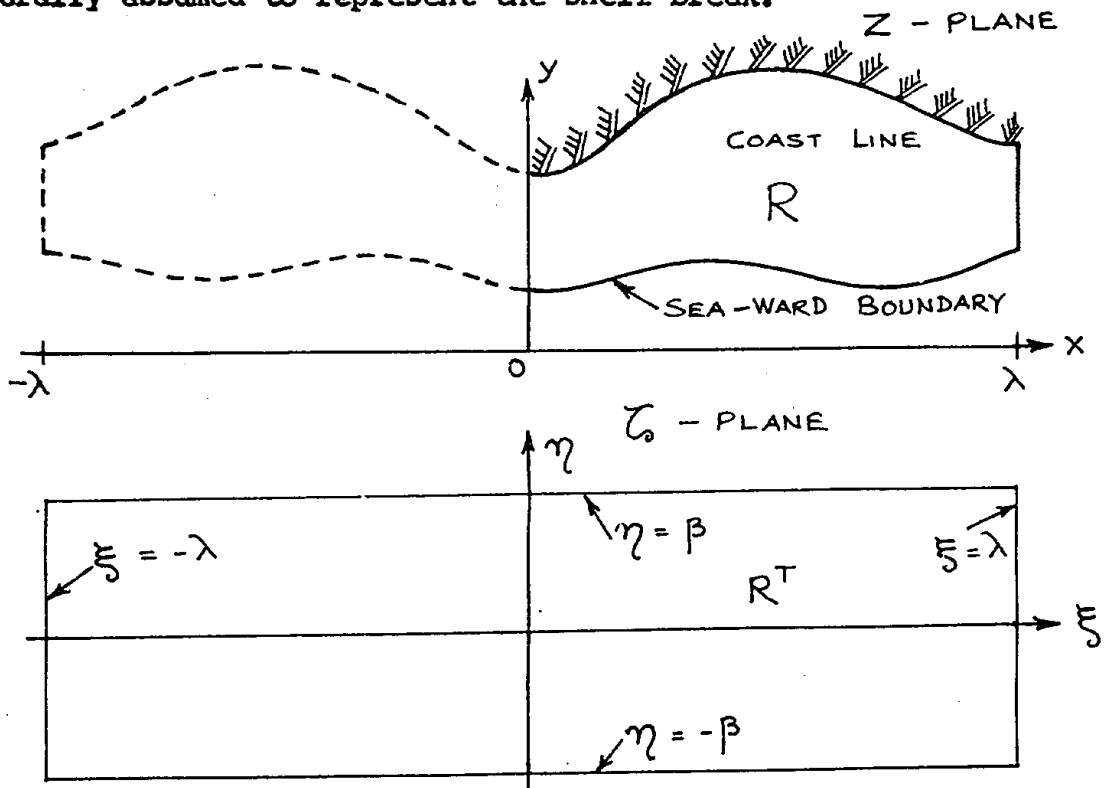


Figure 1. A typical shelf region in the real plane and in the transformed plane.

The problem now is to find a suitable transformation function

$$Z = F(\zeta) \quad (1)$$

where $Z = x + iy$, $\zeta = \xi + i\eta$ and $i = \sqrt{-1}$ such that the coast line and the sea-ward boundary are mapped as isolines of the coordinate η (i.e. the coast line maps on to the isoline $\eta = \beta$, and the sea-ward boundary to the isoline $\eta = -\beta$) and the two lateral boundaries in the Z -plane are mapped as isolines of the coordinate ξ . The functional form considered for the transformation is the truncated Fourier series

$$F(\zeta) = P_0 + Q_0 \zeta + \sum_{n=1}^N (P_n \cos nk\zeta + Q_n \sin nk\zeta) \quad (2)$$

where $k = \pi/\lambda$ and λ is the length of the shelf along the x -direction. The coefficients P_n and Q_n are complex constants of the form

$$\begin{aligned} P_n &= A_n + i B_n \\ Q_n &= C_n + i D_n \quad , n = 0, 1, 2, 3, \dots, N \end{aligned}$$

where A_n, B_n, C_n and D_n are real. Now if we impose the condition that the transformation function is symmetric about y axis, it requires that $A_n = D_n = 0$, for $n = 0, 1, 2, \dots, N$. Also, this condition assures that the lateral boundaries of the mapped region represent straight parallel lines normal to the shore line and the sea-ward boundary. It may be noted here that only the region R , to the right of the y axis, represents the real region of the shelf. Further if we assume that the range for x and ξ is same, we get the relation for x and y in terms of ξ and η as

$$x(\xi, \eta; B_0, B_n, C_n) = \xi + \sum_{n=1}^N (B_n \sinh nk\eta + C_n \cosh nk\eta) \sin nk\xi \quad (3a)$$

$$y(\xi, \eta; B_0, B_n, C_n) = B_0 + \sum_{n=1}^N (B_n \cosh nk\eta + C_n \sinh nk\eta) \cos nk\xi \quad (3b)$$

where B_0 and the set of coefficients B_n and C_n are the unknowns to be determined, such that the equations (3a,b) give the best fit in a least square sense, to the coast line when $\eta = \beta$ and to the sea-ward boundary when $\eta = -\beta$. Here, β is also an additional unknown parameter of this problem. The problem now reduces to the minimization of the following error function

$$E = W_{xc} E_{xc} + W_{yc} E_{yc} + W_{xs} E_{xs} + W_{ys} E_{ys} \quad (4)$$

where

$$\begin{aligned}
 E_{xc} &= \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \left[X_c(\xi) - x(\xi, \beta; B_0, B_n, C_n) \right]^2 d\xi \\
 E_{yc} &= \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \left[Y_c(\xi) - y(\xi, \beta; B_0, B_n, C_n) \right]^2 d\xi, \quad (5a, b, c, d) \\
 E_{xs} &= \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \left[X_s(\xi) - x(\xi, -\beta; B_0, B_n, C_n) \right]^2 d\xi, \\
 E_{ys} &= \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \left[Y_s(\xi) - y(\xi, -\beta; B_0, B_n, C_n) \right]^2 d\xi,
 \end{aligned}$$

and W_{xc} , W_{yc} , W_{xs} , W_{ys} are the weight factors. X_c , Y_c and X_s , Y_s are the coordinates respectively of the coastline ($\eta = \beta$) and the sea-ward boundary ($\eta = -\beta$) of the specified shelf region. $x(\xi, \beta; B_0, B_n, C_n)$ and $y(\xi, \beta; B_0, B_n, C_n)$ are the transform generated coordinates of the shoreline. Similarly $x(\xi, -\beta; B_0, B_n, C_n)$ and $y(\xi, -\beta; B_0, B_n, C_n)$ are the transform generated coordinates of the sea-ward boundary. Obviously, this minimization problem has no direct procedure for the determination of β and other coefficients. Wanstrath et al (1976) has suggested an interactive scheme which, in each iteration computes the unknown coefficients in two steps. It has to be pointed out here that Wanstrath et al (1976) introduced weighting factors, perhaps, for accelerating the convergence of the iterative scheme, the weighting factors being estimated from the errors in the preceding iteration. But, when this procedure of Wanstrath et al (1976) was used, we found that the convergence of the iterative scheme was not improved. Hence, we have taken the weighting factors as unity in our computations.

In the first step B_0 and β , appearing as the first two terms of $y(\xi, \beta; B_0, B_n, C_n)$ and $y(\xi, -\beta; B_0, B_n, C_n)$ are treated as unknowns, and when the error function (equation 4) is minimized, we get for the $(1 + 1)$ th iteration

$$B_0^{1+1} = (I_1 + I_2) / 2\lambda, \quad (6a)$$

$$\text{and } \beta^{1+1} = (I_1 - I_2) / 2\lambda, \quad (6b)$$

$$\text{where } I_1 = \int_{-\lambda}^{\lambda} Y_c(\xi) d\xi \quad \text{and } I_2 = \int_{-\lambda}^{\lambda} Y_s(\xi) d\xi$$

In the next step, with these values of B_0 and β , B_n and C_n are

determined by again minimizing the error function. Thus we have

$$B_n^{1+1} = \frac{2}{\lambda \cosh 2nk \beta^{1+1}} \left\{ (I_3 - I_4) \sinh nk \beta^{1+1} + (I_5 - I_6) \cosh nk \beta^{1+1} \right\} \quad (7a,b)$$

$$C_n^{1+1} = \frac{2}{\lambda \cosh 2nk \beta^{1+1}} \left\{ (I_3 + I_4) \cosh nk \beta^{1+1} + (I_5 - I_6) \sinh nk \beta^{1+1} \right\}$$

$$I_3 = \int_0^\lambda \left[X_C(\xi) - \xi \right] \sin nk \xi \, d\xi$$

$$I_4 = \int_0^\lambda \left[X_S(\xi) - \xi \right] \sin nk \xi \, d\xi$$

$$I_5 = \int_0^\lambda Y_C(\xi) \cos nk \xi \, d\xi, \text{ and}$$

$$I_6 = \int_0^\lambda Y_S(\xi) \cos nk \xi \, d\xi.$$

The iterative procedure is initiated by setting all B_n and C_n to zero and all weight factors to unity. For succeeding iterations B_n and C_n values are taken from the preceding iterations. It is to be noted here that X_C , Y_C and X_S and Y_S in equations (5), (6) and (7) are not known directly in terms of ξ . In order to express them as functions of ξ the given boundary curves and the transform generated boundary curves at equal intervals of ξ , are first expressed parametrically in terms of the arc lengths measured along each curve from some fixed point. But the lengths of the transform generated curve and the given curve will not generally be equal unless the generated curve matches exactly with the given curve. Hence the arc lengths along the generated curve is multiplied by a suitable factor so that the two curves have exactly the same length. Expressing X_S , Y_S , X_C and Y_C as functions of ξ at equal intervals of $\delta\xi$ between 0 and λ , Wanstrath *et al* (1976) evaluated the integrals I_1 to I_6 by using the trapezoidal rule. The use of trapezoidal rule implies that the integrands can be approximated by linear functions within small intervals $\delta\xi$. This might introduce errors in the evaluation of the integrals I_3 to I_6 at large values of n due to the presence of $\frac{\cos}{\sin}(nk\xi)$ terms in the integrands. Hence we used an alternative scheme in which exact integrals are first computed over small intervals of ξ

assuming that X_c , Y_c , X_s and Y_s are linear functions of ξ over these intervals. For example (see figure 2), the contribution to the integral I_3 from the j th interval is given by

$$\begin{aligned}
 I_{3,j} &= \int_{\xi_j}^{\xi_{j+1}} \left[X_c(\xi) - \xi \right] \sin nk\xi \, d\xi \\
 &= \int_{\xi_j}^{\xi_{j+1}} \left[A+B\xi \right] \sin nk\xi \, d\xi \\
 &= (\cos nk\xi_j - \cos nk\xi_{j+1}) A/nk + (\xi_j \cos nk\xi_j \\
 &\quad - \xi_{j+1} \cos nk\xi_{j+1}) B/nk + (\sin nk\xi_{j+1} \\
 &\quad - \sin nk\xi_j) B/\eta^2 k^2
 \end{aligned}$$

where $B = \frac{(X_{c,j+1} - X_{c,j})}{\delta\xi} - 1$, and $A = X_{c,j} - B\xi_j$

The complete integral I_3 is now evaluated by adding the contribution from all the intervals. Similarly the other integrals are also evaluated.

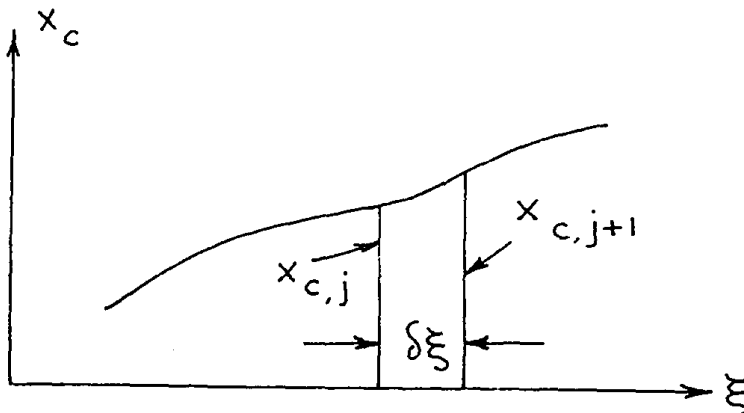


Figure 2. X_c as a function of ξ .

3. MAPPING OF THE SHELF REGION ALONG THE EAST COAST OF INDIA

It has been observed that 75% of the storms that struck the Indian coast in the last hundred years, developed in the Bay of Bengal and moved towards the east coast of India. Hence, two overlapping stretches of the shelf region along the east coast frequently experiencing storm surges, have been chosen and the transformation coefficients were computed to map these areas into rectangles. The seaward boundary of this shelf region is taken along the 200-meter depth contour. The coast line and the 200-meter depth contour are smoothed subjectively before reading them from the hydrographic charts to suppress small scale variations. For example narrow river entrances are replaced by fictitious coastline across the entrance.

The two areas selected are the shelf region (see figure 3)

- (A) between Madras and Kalingapatnam, and
- (B) between Porto Novo and Visakhapatnam.

The coefficients of the transformation function to conformally map the specified region (A) and (B) into rectangles, were computed through a computer program developed for TDC 316 (see Fernandes and Mahadevan, 1983 for the computer program). The transform generated boundaries of the two regions are plotted in figures (4) and (5) respectively. B_0 , β and the mean square error E between the specified and the transform generated coordinates for region (B) are presented in table (1). It can be seen that E fails to converge beyond 7 iterations. For better agreement between the specified boundaries and the transform generated boundaries, more number of coefficients have to be considered as has been observed by Wanstrath et al (1976) who computed 160 coefficients. Computation of 20 transformation coefficients through 20 iterations with 50 integration steps in the interval $0 \leq \xi \leq \lambda$, required 60 minutes of machine time on TDC 316. Hence we did not attempt to compute more than 20 coefficients because of the large computer time involved. However, the closeness obtained between the specified and the generated boundaries using less number of transformation coefficients, is sufficient for storm surge computations when compared to stair-step approximations normally made for representing the boundaries in such problems.

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Table 1. Convergence* table for region B.

No. of Iterations	E (Km)	B_0 (Km)	β (Km)
1	83.06	109.8	19.6
2	53.04	110.3	15.1
3	43.22	110.9	15.3
4	37.67	111.5	14.9
5	35.14	111.9	14.7
6	34.10	112.2	14.5
7	33.75	112.4	14.4
8	33.76	112.6	14.3
9	34.02	112.6	14.3
10	34.31	112.7	14.2

* 10 coefficients were computed with 50 integration steps in the interval $0 \leq \lambda \leq \lambda$ ($\lambda = 770$ Km).

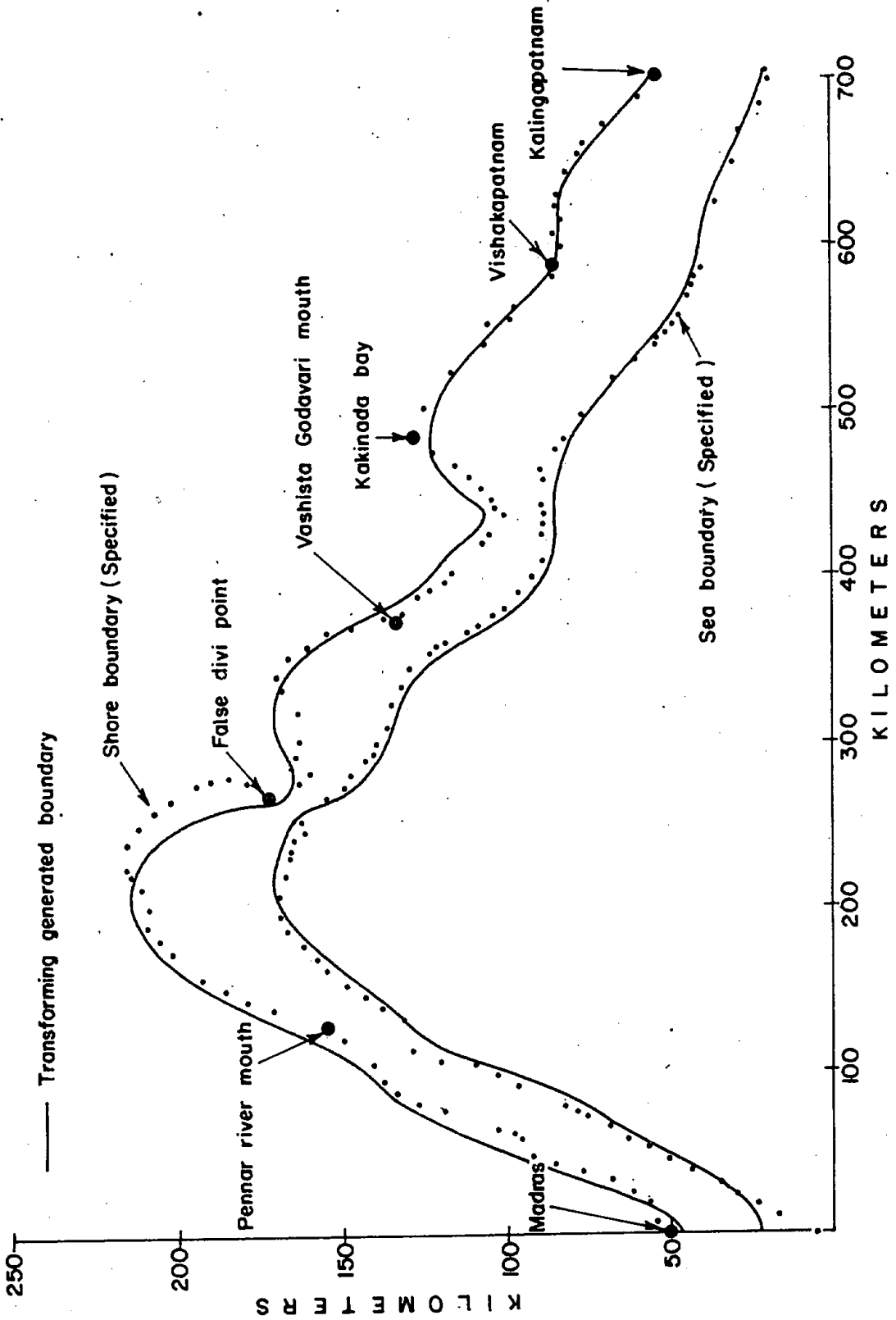


Fig. 4. SHELF REGION BETWEEN MADRAS AND KALINGAPATNAM

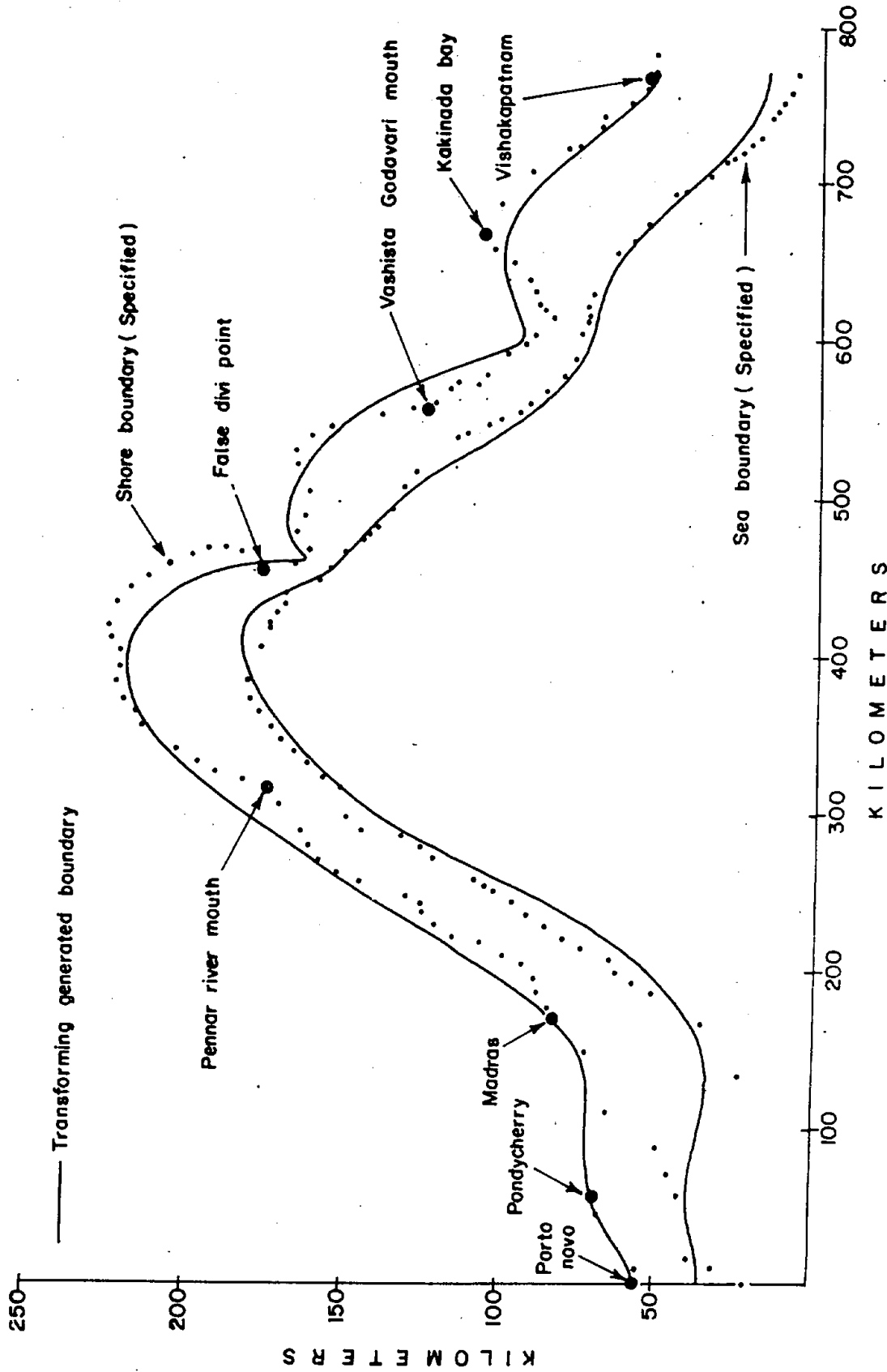


Fig. 5. SHELF REGION BETWEEN PORTO NOVO AND VISHAKAPATNAM